Optimal control of communication in energy constrained sensor networks through team theory and Extended RItz Method

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***Abstract*— The Extended RItz Method (ERIM) can be used to face optimal decision and control problems when finding the global solution is hard, because the problem is ill-conditioned or we can only compute the solution via numerical approximations. It consists in constraining the control functions to take on a fixed structure with a certain number of free parameters to be optimized. We will show the use of such method for the solution of a communication problem in a mixed (analog/digital) transmission environment. A noisy channel is used to convey information from a limited-energy analog device to a sink; in the presence of a binary link, how can we reduce the energy spent for transmission without renouncing reconstruction capability and real-time encoding?**

I. Introduction

It is well known that decentralized optimal control problems involving the interaction between information and control of multiple decisional agents can be very difficult to solve. A classical case is that of Witsenhausen's counterexample [1], where in the presence of two decision makers, a nonlinear optimal solution is known to exist (but has never been found analytically), which outperforms any linear strategy, even under LQG (linear information structure, quadratic cost and Gaussian noise) hypotheses. Ho et al. [2] pointed out the relation of a specific formulation of Witsenhausen's problem to a classical information theoretic one, whose structure, on the other hand, admits a linear optimal solution (which, however, is not found by an optimization-theoretic approach, but rather using the information theoretic concepts of rate-distortion and channel capacity) - namely, the energy-constrained transmission of a Gaussian source over a Gaussian additive noise channel with quadratic distortion measure (the so-called "Gaussian test channel"). It is also worth noting that in this case the two decision makers play the role of an encoder and a decoder, respectively, which turn out to be, in an information theoretic sense, "instantaneous" or "single letter" (i.e., the minimum distortion is not achieved asymptotically over long sequences of source symbols; the reasons for this behavior have recently been further investigated by Gastpar et al. [3]). It was shown by Bansal and Başar [4] that the presence in the cost function of Witsenhausen's problem of a product term between the decision variables gives rise to nonlinearity in the optimal solution. In this case, many suboptimal solutions have been sought in the literature. In Baglietto et al. [5], the optimal decision strategies are approximated by means of fixed-structure parametrized nonlinear functions, by using the Extended RItz Method (ERIM) [6]. The short discussion above shows how even small variations in the problem formulation in a decentralized decision theoretic framework can produce very different responses. A similar case is considered in the present paper, where we deal with a communication problem in the presence of a mixed channel environment. More specifically, our setting is exactly the same as the Gaussian test channel, but with the addition of a parallel binary channel, creating the possibility of a second "description" between the encoder and the decoder. Therefore, we are no longer under the hypotheses leading to a linear structure and instantaneous coding strategies. However, we want to keep the latter characteristic as a constraint, and we want to investigate the impact of strategies where the nonlinearity is reduced to a simple on-off decision, and the binary channel is used as a signaling one to communicate the outcome of this decision to the decoder. In order to highlight the effect of the presence of the binary channel, we assume it to be noiseless.[1] The rationale for considering such on-off signaling strategies arises, for instance, in the collection of measurements in a sensor network, where each element can decide weather to transmit or not the observed realization of a phenomenon described by a Gaussian random variable to a sink, according to the "significance" of the measured value. Refraining from transmission results in lower energy consumption. The case of the Gaussian vector channel [8] - where, interestingly enough, the best linear solution turns out to be also instantaneous and may prevent some components with low signal-to-noise ratio to be transmitted, in favour of others - will be the subject of future work: in this paper we only refer to the single sensor - sink case, which is the simplest energy-constrained communication problem of this kind. A suboptimal solution for this kind of problem was originally considered in [7], where the space of admissible strategies was constrained in order to obtain a closed-form solution. It is worth noting that "hybrid" (analog/digital) communication schemes like the one in our setting are actually possible and have been considered in the communications literature (see, e.g. [9],[10]).

Here we will develop a more precise approximation to the optimal (constrained) strategies. In order to find better solutions, as close as possible to the global optima, we suggest to use a functional approximation technique: after the encouraging results of Baglietto et al. [5] in the solution

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[1]The presence of a noisy BSC (Binary Symmetric Channel) can be taken into account without substantial modifications of our approach.

of Witsenhausen's counterexample, the ERIM was chosen, being a well-known approach for the solution of functional optimization problems [6][11]. The use of ERIM combined with team theory consents, with a finite number of parameters, to cover a wider space of admissible solutions, which is only limited by the smoothness hypotheses of the functions to be approximated.

The paper is organized as follows. The precise problem formulation is reported in Section II, where the suboptimal solution of [7] is also recalled. Section III describes the application of ERIM to the problem. Special cases and numerical results are discussed in Sections IV and V. Section VI contains the conclusions.

II. PROBLEM FORMULATION

Consider the problem in Fig. 1, where the decision maker DM_1 has to decide whether or not to transmit the measured variable $x = y_1$ (or its elaboration) over the analog channel, perturbed by noise v, and the decision maker DM_2 must compute the best estimate u_2 of the measured variable x, according to the analog noisy signal y_{22} and the binary signal ($y_{21} = u_{11} = 1$ means transmission, while $y_{21} = u_{11} = 0$ means no transmission) that it receives perfectly from DM_1. To resume, the channel communication equations are:

$$y_{21} = u_{11} \in \{0,1\} \ , \ y_{22} = u_{12} + v \in \mathbb{R} \tag{1}$$

and we define $\mathbf{u}_1 = col(u_{11}, u_{12})$ and $\mathbf{y}_2 = col(y_{21}, y_{22})$. The stochastic variables are gaussian, with zero mean and known variance:

$$x \sim N(0, \sigma_x^2) \ , \ v \sim N(0, \sigma_v^2) \ . \tag{2}$$

The transmission of DM_1 is subject to a power constraint:

$$\mathop{E}_{x}\left\{u_{12}^2\right\} \leq 1 \ . \tag{3}$$

The two decision makers cooperate in order to minimize the average distortion:

$$d = E\left\{J_D\right\} = E\left\{(y_1 - u_2)^2\right\} \ . \tag{4}$$

Remark 1: The information vector of the two decision makers is: $I_1 = y_1, \ I_2 = \mathbf{y}_2$.
The goal of the communication control problem is to find the optimal functions, γ_1, γ_2, which implement the two decision makers DM_1, DM_2, such that the average distortion in the communication process is minimum, and the source does not exceed its physical limit in transmission power.

Remark 2: If $\sigma_x^2 \leq 1$ and $u_{12} = x$, then the constraint (3) is always honored.

Problem 1: For the communication problem in Fig. 1, where (1) and (2) hold, find the optimal decision functions $\gamma_1 : \mathbb{R} \mapsto \{0,1\} \times \mathbb{R}$, $\gamma_2 : \{0,1\} \times \mathbb{R} \mapsto \mathbb{R}$:

$$\mathbf{u}_1 = \gamma_1(y_1) \tag{5}$$
$$u_2 = \gamma_2(\mathbf{y}_2) \tag{6}$$

subject to constraint (3), so that the average distortion (4) of the communication is minimum.

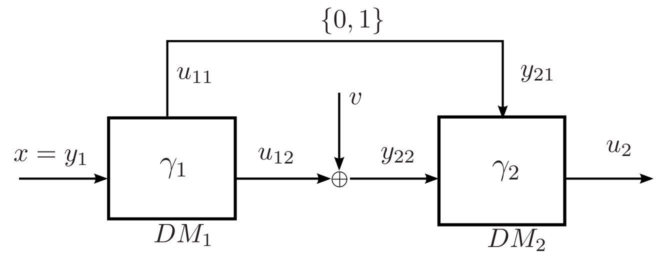


Fig. 1. The basic communication problem (encoder/decoder - sensor/sink).

A. Suboptimal analytical solution

Problem 1 suffers from the same difficulties which have been investigated by Ho et al. [2], in particular the solution of a complex functional optimization problem. Aware of the issues, Davoli in [7] gives up searching for global solutions and instead considers a modified version of the problem, which allows the analytical computation of a suboptimal solution. To be precise, the following assumptions are made.

Assumption 1: The structure of DM_1 is partially constrained: $\gamma_{12}(y_1) = y_1\gamma_{11}(y_1)$.
In other words, the encoder's decision is on whether to send the observed value or not, and the binary channel is used to communicate this decision to the decoder.

Assumption 2: DM_2 ignores the shape of $\gamma_{11}(\cdot) \triangleq \varphi(\cdot)$, in particular the following conditional probability cannot be computed: $p_{y_{21}|y_1}(u_{11}|X)$.
By applying the *person-by-person optimality* principle of team theory [12], finding the optimal solution to Problem 1, under Assumptions 1 and 2, can be partitioned into two subproblems[2]:

$$\min_{\gamma_{11}} E\left\{(y_1 - u_2)^2|\gamma_2^*\right\} \tag{7}$$
$$\min_{\gamma_2} E\left\{(y_1 - u_2)^2|\gamma_{11}^*\right\} \tag{8}$$

which have to be solved, respectively, for the encoder DM_1 and the decoder DM_2 (and in the following will be named "the encoder/decoder problem").

Remark 3: In the two subproblems, γ_{12} is fixed once γ_{11} is known, as in Ass. 1; hence, there is no need to condition the expected values with respect to γ_{12}.
In [7], under the condition $\sigma_x^2 = 1$, the optimal decision function γ_2° is found after the solution of (7). Afterward, in virtue of Ass. 2, γ_{11}° is computed analytically; then, applying Ass. 1, γ_{12}° is found. More precisely, the decoder problem yields

$$u_2^\circ = \gamma_2^\circ(\mathbf{y}_2) = y_{21} \ \frac{1}{1+\sigma_v^2} \ y_{22} \triangleq y_{21} \ \lambda \ y_{22} \tag{9}$$

while the subsequent encoder problem yields

$$u_{11}^\circ = \gamma_{11}^\circ(y_1) = \begin{cases} 0, & |y_1| \leq \sqrt{\frac{\sigma_v^2}{1+2\sigma_v^2}} \triangleq \alpha \\ 1, & \text{otherwise} \end{cases} \tag{10}$$

[2] γ^* means that the function γ is fixed.

Then, applying Assumption 1 we can compute γ_{12}°. The strategies (10) and (9) constitute a couple of individual optimal strategies, or "person-by-person" optimal strategies.It is obvious that the two assumptions lead to a suboptimal solution of Problem 1. Nevertheless, it can be proved that the average distortion produced by the suboptimal strategies, $\bar{d}^{\circ}$, is always lower then the one produced by the optimal encoder-decoder couple in the absence of the binary channel, corresponding to Shannon's optimal linear filters, $\bar{d}_{\text{Sh}} = \sigma_v^2/(1+\sigma_v^2)$. Some numerical results, proving the inequality, are reported in Table I: for different signal-to-noise ratios (keeping $\sigma_x^2 = 1$ and varying σ_v^2) we computed the α parameter of γ_{11} (called the "dead-zone" parameter, because it delimits the no-transmission interval), Shannon's distortion limit for the Gaussian source over the Gaussian channel, and the numerical approximation of transmission power and distortion, corresponding to the application of (9) and (10) on $N \cong 10^7$ different random realizations $(x,v) = (x(k), v(k))\,, k = 0,\dots,N-1$. In a practical way, this shows that the contribution of the binary channel "enhances" the communication process. In order to find a better solution (as close as possible to the global optima) to Problem 1 than the one found in [7], as mentioned before we will use a functional approximation technique, more precisely the ERIM. In the following, we report the details of the procedure which is necessary to apply the ERIM to find the numerical approximation of the global optimal solution of Problem 1: specifically, the back-propagation technique applied to the dynamic structure of the team and the capability to handle binary signals and particularly difficult stochastic constraints. For further details on the ERIM, we refer to [6].

Remark 4: For the simplicity and economy of implementation, a linear suboptimal solution strongly discourages the resource-demanding search of the global optimal nonlinear solution. In fact, in this paper a technique is proposed, which consents the approximation of the global optimal solution, by means of neural approximators. Their use allows to find a global cost that is inferior to the linear solution one, and to keep the solution easy to implement (even in hardware).

σ_v^2	$\bar{d}_{\text{Sh}}$	$\bar{d}^{\circ}$	α	P
0.0	0	0	0	1
0.001	0.00099900	0.000010	0.031591	0.999474
0.01	0.00990099	0.000446	0.099015	0.999224
0.03	0.02912621	0.002838	0.168232	0.998225
0.1	0.09	0.020827	0.288675	0.993247
1.0	0.5	0.425461	0.577350	0.953148

TABLE I

PROBLEM 1: SHANNON'S DISTORTION LIMIT, DAVOLI'S DISTORTION COST, AVERAGE TRANSMISSION POWER, AND PARAMETER α FOR THE SUBOPTIMAL STRATEGIES, WITH RESPECT TO σ_v^2 ($\sigma_x^2 = 1$).

III. SOLUTION OF THE COMMUNICATION PROBLEM BY MEANS OF THE ERIM

In order to solve Problem 1, we shall give our functions a fixed structure. As to the binary signal u_{11}, we shall generate it by means of a continuous quantity $z = \gamma_{11}(y_1)$ and a 'sign' function, i.e. $u_{11} = \text{sgn}(z)$.

The generation of signal u_{12} will be obtained as $u_{12} = u_{11}\gamma_{12}(y_1)$. This implies that the transmission of $\gamma_{12}(y_1)$ (that is, the elaboration of the measured variable y_1) occurs only when the binary signal is $u_{11} = 1$; if $u_{11} = 0$, there is no transmission ($u_{12} = 0$), and the energy for transmission is saved. The graphical representation of the generation of $\mathbf{u}_1$ is shown in Fig. 2. Within the new assumptions, we can reformulate Problem 1 as:

Problem 2: For the communication problem in Fig. 1, where u_{11} is generated as in Fig. 2, and (1), (2) and (6) hold, find the optimal decision functions $\gamma_{11}^{\circ} : \mathbb{R} \mapsto \mathbb{R}$, $\gamma_{12}^{\circ} : \mathbb{R} \mapsto \mathbb{R}$, $\gamma_2^{\circ} : \{0,1\} \times \mathbb{R} \mapsto \mathbb{R}$, subject to constraint (3), so that the average distortion (4) of the communication process is minimum.

Remark 5: From a team theory point of view, $\gamma_{11}, \gamma_{12}, \gamma_2$ can be considered as three agents in a cooperative game.

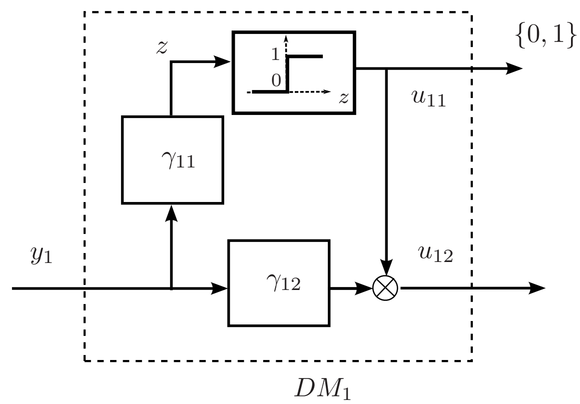


Fig. 2. The generation of $\mathbf{u}_1$ from the encoder, where $u_{11} = \text{sgn}(\gamma_{11}(y_1))$ and $u_{12} = u_{11}\gamma_{12}(y_1)$.

The solution of Problem 2 is not easy, because:

- it is an infinite-dimension functional optimization problem (as the goal is to find optimal functions);
- the information structure $I_2 = \mathbf{y}_2$ is not well-defined until the strategies γ_{11}, γ_{12} are fixed; that is the probabilities which are required for the computation of the solution depend on the solution itself: the problem of variable estimation is no longer separable from the control and decision one;
- the optimization problem $\min\limits_{\gamma_{11},\gamma_{12},\gamma_2} J(\gamma_{11}, \gamma_{12}, \gamma_2)$ is difficult to solve because of functional dependencies, i.e. $\gamma_2 = \gamma_2(\gamma_{11}, \gamma_{12})$.

These difficulties are characteristic of team decision problems with dynamic information structures, and have been widely investigated since the well known Witsenhausen's counterexample [1]. Following the successful approach of Baglietto et al. [5] for the solution of the aforementioned problem, we shall apply the ERIM in order to solve Problem 2. The admissible decision functions $\gamma_{11}, \gamma_{12}, \gamma_2$ are constrained to take on a fixed but parametrized structure, indeed in the form of one hidden layer neural networks:

$$\hat{\gamma}_j(z, \mathbf{w}_j) = \sum_{i=1}^{\nu} c_{ij}\varphi(z, \kappa_i) + b_j \ , \ j = 11, 12, 2 \quad (11)$$

where $\mathbf{w}_j = [\{c_{ij}\}, \{\kappa_i\}, b_j]$, $c_{ij}, b_j \in \mathbb{R}$; moreover, if $j = 11, 12$, $\kappa_i \in \mathbb{R}^2$ and $z = y_1$; if $j = 2$, $\kappa_i \in \mathbb{R}^3$ and $z = \mathbf{y}_2 = col(y_{21}, y_{22})$. The hidden layer function φ can be a radial basis function, a sigmoidal one, etc. In our case, we choose sigmoidal functions, $\varphi : \mathbb{R} \mapsto [-1, 1]$. By substituting the control functions with the parametrized ones (11), the functional cost $J(\gamma_{11}, \gamma_{12}, \gamma_2)$ is turned into a function $J(\mathbf{w}_{11}, \mathbf{w}_{12}, \mathbf{w}_2)$, which is dependent on a finite number of parameters[3], namely $\mathbf{w} \triangleq col(\mathbf{w}_{11}, \mathbf{w}_{12}, \mathbf{w}_2)$, where $dim(\mathbf{w}) = 3(2\nu + \nu + 1) = 9\nu + 3$, if the three neural networks have the same number ν of "neurons" (we remind that ν is related to the approximation capability of the neural network [13]). We can now restate Problem 2.

Problem 3: For the communication problem in Fig. 1, where u_{11} is generated as in Fig. 2, and (1), (2) and (6) hold, find the vectors of optimal parameters $\mathbf{w}_{11}^{\circ}, \mathbf{w}_{12}^{\circ}, \mathbf{w}_2^{\circ}$, being

$$\begin{aligned} u_{11} = &\operatorname{sgn}(\hat{\gamma}_{11}(y_1, \mathbf{w}_{11})), \ u_{12} = u_{11}\hat{\gamma}_{12}(y_1, \mathbf{w}_{12}), \\ &u_2 = \hat{\gamma}_2(\mathbf{y}_2, \mathbf{w}_2) \end{aligned} \quad (12)$$

where $\hat{\gamma}_{11} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_{12} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_2 : \{0, 1\} \times \mathbb{R} \mapsto \mathbb{R}$ have the structure (11), and are subject to constraint (3), so that the average distortion (4) of the communication is minimum.

Problem 3 is a constrained non-linear programming problem; it is possible to re-interpret the hard constraint (3), which delimits the set of admissible values $\mathbf{w} = col(\mathbf{w}_{11}, \mathbf{w}_{12}, \mathbf{w}_2) \in W \subseteq \mathbb{R}^{3\nu}$, as a soft constraint, by means of a penalty function, which can be added to the cost function to be minimized. We can define the penalty cost term J_P:

$$J_P = \kappa_P f(\bar{P}) = \kappa_P \left[\max(\bar{P} - 1, 0)\right]^2 \quad (13)$$

where $\bar{P}$ is the average transmission power. Obviously, only a numerical approximation of $\bar{P}$ (over N successive realizations of the random variable x) can be computed:

$$\bar{P} \triangleq \mathop{E}_{x}\left\{u_{12}^2\right\} \cong \frac{1}{N}\sum_{i=0}^{N-1}\left[u_{12}^{(i)}\right]^2 \quad (14)$$

Remark 6: The correct choice of κ_P is crucial. It weights the excess of power limit, that is the violation of the original constraint. It is important to balance it with the cost to be minimized (the distortion), above all during the "training" of the networks. For a dissertation on the optimal choice of this parameter for quadratic penalty functions, see [14].
Then we can restate Problem 3.

Problem 4: For the communication problem in Fig. 1, where u_{11} is generated as in Fig. 2, and (1), (2), (6) and (12) hold, where $\hat{\gamma}_{11} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_{12} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_2 : \{0, 1\} \times \mathbb{R} \mapsto \mathbb{R}$ have the structure (11), find the vectors of optimal parameters $\mathbf{w}_{11}^{\circ}, \mathbf{w}_{12}^{\circ}, \mathbf{w}_2^{\circ}$, so that the cost

$$\bar{J}(\mathbf{w}) = \mathop{E}_{x,v}\left\{J_D + J_P\right\} = \mathop{E}_{x,v}\left\{(x - u_2)^2 + \kappa_P f(\bar{P})\right\} \quad (15)$$

of the communication process is minimum.
Problem 4 is now an unconstrained nonlinear programming problem, which can be solved by a usual gradient descent method, i.e.

$$\begin{aligned} \mathbf{w}_j(k+1) = \mathbf{w}_j(k) - s(k)\nabla_{\mathbf{w}_j}\bar{J}(\mathbf{w}(k)) + \\ + \eta(\mathbf{w}_j(k) - \mathbf{w}_j(k-1)), \ j = 11, 12, 2 \end{aligned} \quad (16)$$

where k is the generic iteration step ($k = 0, 1, \ldots, K-1$; K can be defined a priori or be consequent to a stop condition), $s(k)$ a suitable step-size, and $\eta \in [0, 1]$ a constant weighting a regularization term. To be more precise, the application of algorithm (16) requires the following assumption:

Assumption 3: $\bar{J}(\mathbf{w})$ from (15) is C^1 (first order continuity) with respect to $\mathbf{w}$.
If Ass. 3 holds, then under certain regularity hypotheses, $\bar{J}(\mathbf{w}(k)) = \mathop{E}_{x,v}\left\{J(\mathbf{w}(k))\right\}$ is also continuous and differentiable, and:

$$\nabla_{\mathbf{w}} \mathop{E}_{x,v} J(\mathbf{w}(k)) = \mathop{E}_{x,v} \nabla_{\mathbf{w}(k)}\left[J(\mathbf{w}(k))\right] \quad (17)$$

Unfortunately, it is practically impossible to compute the gradient analytically or exactly, as a consequence of the stochastic nature of the problem, and because of the computational complexity. Therefore, we will opt for a stochastic approximation technique: a single "realization" $\nabla_{\mathbf{w}(k)}J(\mathbf{w}(k))\big|_{x=x(k),v=v(k)}$ is computed, where the stochastic variables $x(k), v(k)$ are generated randomly according to their known probability density functions. In fact, in this case a "single" couple $(x(k), v(k))$ is not sufficient to compute the gradient, as a consequence of the cost function J, in particular of the penalty term J_P, which requires the computation of the average power $\bar{P}(k)$:

$$\bar{P}(k) \cong \frac{1}{N}\sum_{i=0}^{N-1}\left[u_{12}^i(k)\right]^2 \quad (18)$$

Remark 7: The difference between (14) and (18) is merely formal. In (18) it is emphasized that at each iteration k, the average power is computed for the current vector of parameters $\mathbf{w}(k)$.
To be clear, for a single iteration the average global cost function $\bar{J}(\mathbf{w}(k)) = \bar{J}(k)$ is:

$$\begin{aligned} \bar{J}(k) = \mathop{E}_{x,v}\left\{J_D(k) + J_P(k)\right\} = \mathop{E}_{x,v}\left\{(x(k) - u_2(k))^2\right\} + \\ + \kappa_P \ \max\left[\left(\frac{1}{N}\sum_{i=0}^{N-1}(u_{12}^{(i)}(k))^2 - 1\right), 0\right]^2 \end{aligned}$$

where the following stochastic sequences are generated, according to their known probability functions: $x(k)$,$v(k)$,

[3] The use of parametrized nonlinear approximators avoids the exponential growth of the number of parameters, i.e. incurring in the curse of dimensionality.

$\{x^0(k), \ldots, x^{N-1}(k)\}$. Then a simple gradient steepest descent algorithm can be applied:

$$\mathbf{w}_j(k+1) = \mathbf{w}_j(k) - s(k)\nabla_{\mathbf{w}_j}\bar{J}(k) + \\ + \eta(\mathbf{w}_j(k) - \mathbf{w}_j(k-1)),\ j = 11, 12, 2 \quad (19)$$

for a generic iteration step k. The convergence of the method, which is known as *stochastic gradient* approximation, is assured by a particular choice of the step-size $s(k)$, that must fulfill a set of conditions [15]. Of course, in order to apply the algorithm and find the optimal parameters, one has to compute the partial derivatives of the cost $\bar{J}(k)$ with respect to the parameters to be optimized, $\mathbf{w}_{11}(k), \mathbf{w}_{12}(k), \mathbf{w}_2(k)$:

$$\frac{\partial \bar{J}(k)}{\partial \mathbf{w}_{11}(k)} = \frac{\partial \bar{J}(k)}{\partial u_{11}(k)} \frac{\partial\, \mathrm{sgn}(\hat{\gamma}_{11}(y_1, \mathbf{w}_{11}(k)))}{\partial \mathbf{w}_{11}(k)}, \quad (20)$$

$$\frac{\partial \bar{J}(k)}{\partial \mathbf{w}_{12}(k)} = \frac{\partial \bar{J}(k)}{\partial u_{12}(k)} \frac{\partial u_{11}(k)\hat{\gamma}_{12}(y_1, \mathbf{w}_{12}(k))}{\partial \mathbf{w}_{12}(k)}, \quad (21)$$

$$\frac{\partial \bar{J}(k)}{\partial \mathbf{w}_2(k)} = \frac{\partial \bar{J}(k)}{\partial u_2(k)} \frac{\partial \hat{\gamma}_2(\mathbf{y}_2, \mathbf{w}_2(k))}{\partial \mathbf{w}_2(k)}. \quad (22)$$

Unfortunately the first derivative cannot be computed because of the non-differentiability of the step function, which is necessary for the creation of the binary signal. In order to consent this computation, we will substitute the $\mathrm{sgn}(\cdot)$ function with a differentiable one, which is designed to fit the step as close as possible, depending on a suitable tuning parameter.

Assumption 4: $u_{11} = \sigma_B(\alpha_B \hat{\gamma}_{11}(y_1))$.

Then $u_{12}(k) = \sigma_B(\alpha_B \hat{\gamma}_{11}(y_1, \mathbf{w}_{11}(k)))\hat{\gamma}_{12}(y_1, \mathbf{w}_{12}(k))$. The continuous sigmoid function $\sigma_B(\alpha_B \cdot), \mathbb{R} \mapsto [0,1]$, with increasing α_B, approximates the Heaviside function and is still differentiable (see Fig. 3). Therefore, by an appropriate choice of α_B, we can use it to generate the binary signal u_{11}, preserving the differentiability in the problem. In fact, (20) becomes:

$$\frac{\partial \bar{J}(k)}{\partial \mathbf{w}_{11}(k)} = \frac{\partial \bar{J}(k)}{\partial u_{11}(k)} \frac{\partial \sigma_B(\alpha_B \hat{\gamma}_{11}(y_1, \mathbf{w}_{11}(k)))}{\partial \mathbf{w}_{11}(k)}\ . \quad (23)$$

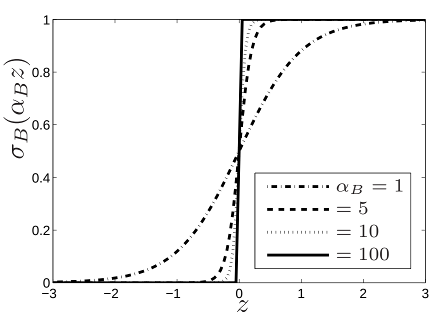


Fig. 3. The shape of $\sigma_B(\alpha_B z)$ from a sigmoid tends to a step-function, with increasing α_B.

For the sake of completeness, we now restate Problem 4 as:

Problem 5: For the communication problem in Fig. 4, under Ass. 4, where (1), (2), (6) and (12) hold, where $\hat{\gamma}_{11} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_{12} : \mathbb{R} \mapsto \mathbb{R}$, $\hat{\gamma}_2 : \{0,1\} \times \mathbb{R} \mapsto \mathbb{R}$ have the structure (11), find the vectors of optimal parameters $\mathbf{w}_{11}^\circ, \mathbf{w}_{12}^\circ, \mathbf{w}_2^\circ$, so that the cost (15) of the communication process is minimum.[4]

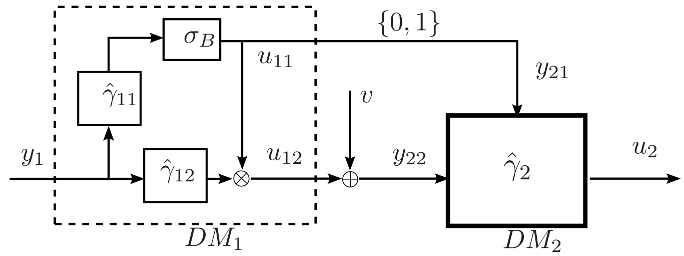


Fig. 4. Problem 5, which can be solved by the ERIM.

The proposed algorithm for the computation of the optimal parameters[5] consists in two phases, a forward and a backward one, and in a back-propagation technique. In the *forward phase* we follow the precedence diagram of the team, and forward all the signals, simulating a communication process. At iteration step k, when the parameters are $\mathbf{w}_{11}(k), \mathbf{w}_{12}(k), \mathbf{w}_2(k)$, a sequence of N couples $(x^i(k), v^i(k))$ is generated, and all the decisions generated by the neural networks are computed; the average power of transmission is estimated. Then, all the partial costs are computed. In the *backward phase*, all the gradient components are computed and eventually "back-propagated" from DM_2 to DM_1. The complete list of equations that are necessary for the computation of (23), (21) and (22), and the consequent application of the stochastic gradient technique, is reported in the Appendix. A pseudo-code description of the algorithm is reported in Alg. 1.

For the numerical simulations performed to train the neural networks we used a particular decreasing step-size, satisfying the convergence conditions: $s(k) = c_1/(c_2 + k)$, with $c_1 \approx 10^2 - 10^4, c_2 \approx 10^7 - 10^9$. Moreover, $\nu = 20$. Training was usually stopped after $K > 10^8$ iterations, or whenever the difference between subsequent weights was properly convergent to zero. To avoid local minima, simulated annealing methods were used. We point out that once the networks are trained, the online phase is very quick, consisting only on a single forward pass (the computational time on a standard Intel x86 Pc is in the order of 10^{-6} seconds).

IV. The Study on the Evenness of γ_{11}

From [7], we know that the suboptimal solution to the communication problem has a certain symmetry, i.e. $\gamma_{11}(y_1)$ is an even function; in particular, it is symmetric with respect to the average of y_1 (which is always zero in our examples, as $E\{x\} = 0$). However, from numerical simulation we found that an unconstrained $\hat{\gamma}_{11}$ yields better results (in term of

[4] We point out that Ass. 4 is necessary only for the "training" phase, as it consents the correct back-propagation of the partial derivatives of $\bar{J}$. Once the optimal parameters are found, in a pure online phase it is not required, and thus the step function can be used (with a further decrease of complexity).

[5] Usually called "training" of the neural networks.

Algorithm 1 Find $\mathbf{w}_{11}^{\circ}, \mathbf{w}_{12}^{\circ}, \mathbf{w}_{2}^{\circ}$ that minimize the average cost $\bar{J}$ of Problem 5

Require: $E\{x\}, \sigma_x^2, E\{v\}, \sigma_v^2, K, N$

Ensure: $\mathbf{w}_{11}^{\circ}, \mathbf{w}_{12}^{\circ}, \mathbf{w}_{2}^{\circ}$

```
for k = 0 to K − 1 do
    compute s(k)
    for i = 0 to N − 1 do
        generate x^i(k), v^i(k) according to their known stochastic properties
        y_1^i(k) ⇐ x^i(k)
        u_11^i(k) ⇐ σ_B(α_B γ̂_11(y_1^i(k), w_11(k))), u_12^i(k) ⇐ u_11^i(k) γ̂_12(y_1^i(k), w_12(k))
        y_21^i(k) ⇐ u_11^i(k), y_22^i(k) ⇐ u_12^i(k) + v^i(k)
        u_2^i(k) ⇐ γ̂_2(y_21^i(k), y_22^i(k), w_2(k))
    end for
    P̄(k) ⇐ (1/N) Σ_{i=0}^{N−1} [u_12^i(k)]^2
    compute (23),(21),(22)
    update w_j(k), j = 11, 12, 2 using (19)
end for
w_j° ⇐ w_j(K), j = 11, 12, 2
```

average distortion) with respect to a $\hat{\gamma}_{11}$, constrained to be even with respect to y_1. Let us compare the two policies[6]:

$$u_{12} = \hat{\gamma}_{12}(y_1, \mathbf{w}_{12}^{\mathrm{u}})\sigma_B(\alpha_B \hat{\gamma}_{11}(y_1, \mathbf{w}_{11}^{\mathrm{u}})) \quad (24)$$

$$u_{12} = \hat{\gamma}_{12}(y_1, \mathbf{w}_{12}^{\mathrm{c}})\sigma_B(\alpha_B \hat{\gamma}_{11}(|y_1|, \mathbf{w}_{11}^{\mathrm{c}})) \quad (25)$$

We trained the networks $\hat{\gamma_{11}}, \hat{\gamma_{12}}$ and $\hat{\gamma}_2$ (all with $\nu = 20$) as described in the previous section, and found different weights for the different policies. We used a modified version of Nguyen's technique to initialize the neural networks weights and biases [16]. Results are shown in Fig. 5, while in Table II the average energy consumption and the average distortion are reported. The fact that the unconstrained solutions do not show any symmetry and outperform the constrained symmetric ones, may be due to a local optimal solution which has been reached during the training. Constraining the neural networks to have an even output, we found another local solution, which attains worse performance with respect to the unconstrained one. These considerations suggest the possibility of improving the optimization phase.

TABLE II

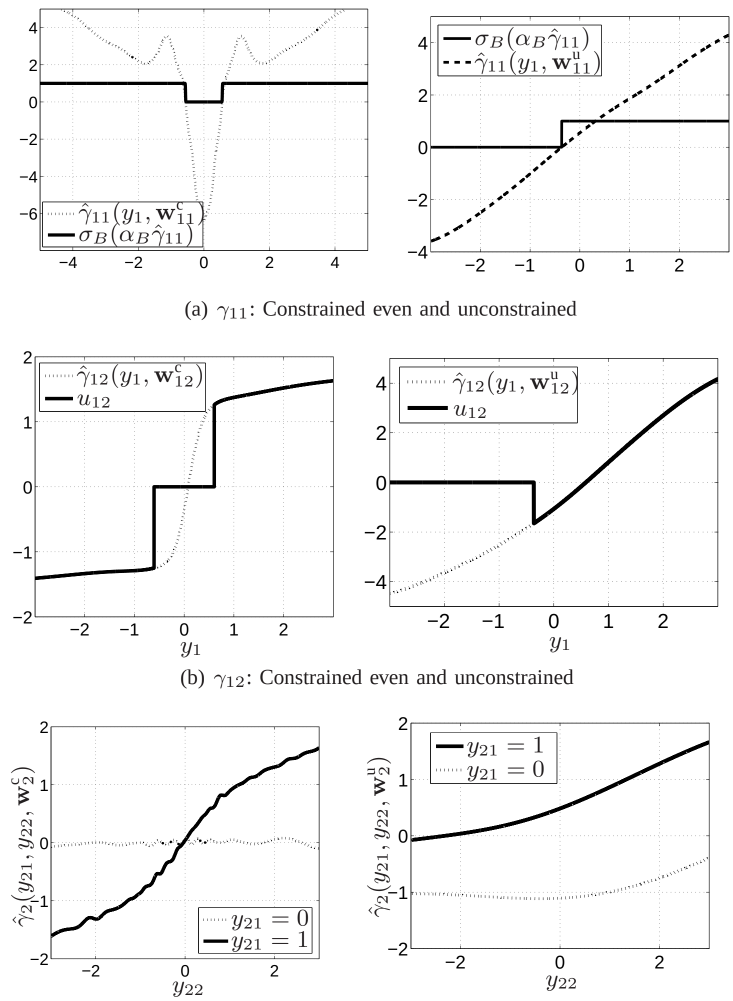
AVERAGE DISTORTION AND ENERGY CONSUMPTION FOR THE POLICIES (24) AND (25), WITH $\sigma_v^2 = 1.0$.

	Constrained even	Unconstrained
$\bar{d}$	0.4051	0.2144
$\bar{P}$	0.9531	0.9923

A. Graphical results

In order to obtain some 'visual' evidence of the difference between the results obtained by (24) and (25), the two

[6]Concerning the weights, u means "unconstrained", while c means "constrained".



(a) γ_{11}: Constrained even and unconstrained

(b) γ_{12}: Constrained even and unconstrained

(c) γ_2: Constrained even and unconstrained

Fig. 5. Encoder and decoder strategies for the policies (25) and (24), with $\sigma_v^2 = 1.0$. When $y_{21} = 0$, u_2 could be found analitically. Nevertheless we report the output of the corresponding neural network for this situation in both cases.

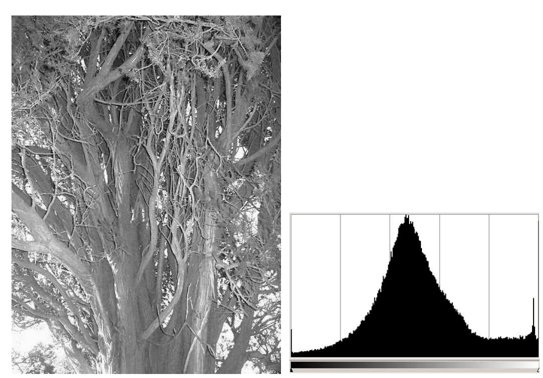
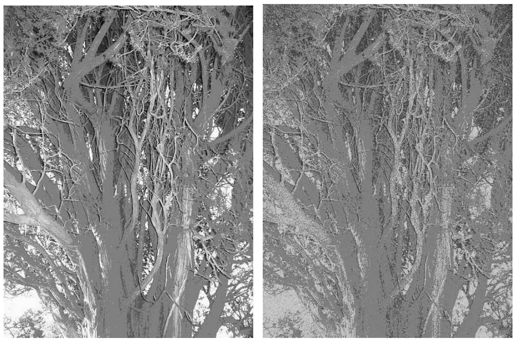


Fig. 6. A gray-scale image, with a gaussian-like histogram.



(a) Constrained even γ_{11}

(b) Unconstrained γ_{11}

Fig. 7. The gray-scale image of Fig. 6 (left) as it is transmitted by the encoder (u_{12}) and (right) the image reconstruction (u_2). In both cases, the gray-scale values have been normalized so that $\sigma_x^2 = 1.0$; the channel noise also has $\sigma_v^2 = 1.0$.

TABLE III

AVERAGE DISTORTION $\bar{d}$ FOR THE IMAGE IN FIG. 6 AFTER THE APPLICATION OF THE POLICIES (24) AND (25).

σ_v^2	Constrained even (a)	Unconstrained (b)
1.0	0.580202	0.437513
0.1	0.011642	0.0102578

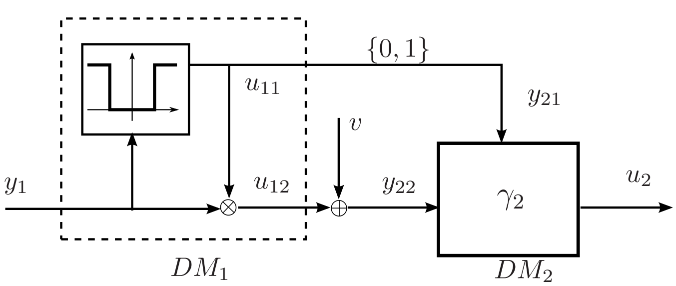


Fig. 8. The "receiver" problem.

σ_v^2	d_s	d_a	P_s	P_a
0.01	0.0004	0.0003	0.9992	0.9977
0.1	0.0208	0.0199	0.9932	0.9931
1.0	0.4254	0.3824	0.9531	0.9518

TABLE IV

"RECEIVER PROBLEM": DISTORTION COST AND AVERAGE TRANSMISSION POWER FOR THE SUBOPTIMAL (S) AND THE NEURAL APPROXIMATED (A) SUBOPTIMAL SOLUTION. ($\sigma_x^2 = 1$).

policies have been applied to the following example: a gray-scale image sent through a noisy analog channel. [7] The image, which is natively a bi-dimensional source, is transformed into a mono-dimensional source by sending one pixel value per time, in raster order (i.e., from the upper left corner of the image to the bottom right corner of the image, line by line from left to right). The image must have a Gaussian-like histogram, i.e., the distribution of values between 0 and 255 has a Gaussian shape (see Fig. 6) [8]; then, the gray values are scaled and shifted from the discrete interval $[0, 255]$ so that the image becomes a Gaussian source with zero mean and unitary variance. Tests were performed for two different channel Gaussian noises, $\sigma_v^2 = 0.1, 1.0$. In Table III average distortions are reported for the two cases. In Fig. 7 the transmitted and estimated images are shown in the case of worst noise. It is evident that the unconstrained policy outperform the constrained one not only from the point of view of the distortion, but from the point of view of the intelligibility of the estimated image.

V. INDIVIDUAL OPTIMAL STRATEGIES AND SUBOPTIMAL SOLUTIONS

In Section II-A suboptimal analytical solutions to Problem 1 have been considered. In the following we shall discuss how we solved the second subproblem (8) (the first, (7) is scarsely interesting, as once fixed DM_2, γ_{11} can be found analytically) through the same method we have described in the previous sections, that has been applied to solve Problem 1.

The reformulation of the "receiver problem", defined by (8) and generating the individual optimal strategy γ_2°, is:

Problem 6: For the communication problem in Fig. 8, under Ass. 4, where (1) and (2) hold, where $\hat{\gamma}_2 : \mathbb{R} \mapsto \mathbb{R}$ has the parametrized structure (11), namely $\hat{\gamma}_2(\mathbf{y}_2, \mathbf{w}_2)$, and

$$\gamma_{12}(y_1) = x\gamma_{11}(y_1) \tag{26}$$

$$\gamma_{11}(y_1) = \begin{cases} 0, & |x| \leq \sqrt{\frac{\sigma_v^2}{1+2\sigma_v^2}} \triangleq \alpha \\ 1, & \text{otherwise} \end{cases} \tag{27}$$

find the vector of optimal parameters $\mathbf{w}_2^\circ$, so that the cost (15) of the communication process is minimum.

The result of the neural training is that whenever a transmission occurs ($y_{11} = 1$), the shape of $\hat{\gamma}_2$ tends to be piecewise linear, with different inclinations; in the other case ($y_{11} = 0$) $\hat{\gamma}_2$ is zero, which is the known average of the stochastic variable x, and the better estimate of x in the case of no transmission, as found before. Some numerical results for the average power and distortion are presented in Table IV.

VI. CONCLUSIONS

In this paper, a neural approach to the optimal control of an energy constrained communication process for a sensor

[7]The use of an image, in this case, is purely explanatory: it is useful to give a qualitative idea of the effects of the two policies.

[8]It is not fundamental to have a "precise" Gaussian histogram, because it is rare to find a natural image with this property (usually to have a "really" Gaussian histogram it is necessary to use a synthetic image).

and sink couple is presented. The optimal control problem is difficult to solve; nevertheless, we can approximate the optimal solution thanks to the well known approximation properties of the Extended RItz Method. The enhancement of the proposed method with respect to a previously addressed analytical solution is significant. In fact, a suboptimal solution could be found analytically only after a sequence of assumptions, which limited the space of admissible strategies. Neural approximators and team theory, with suitable training, can yield better results in the solution of the problem, even in the presence of stochastic constraints and binary signals.

APPENDIX

In the following, the equations which are necessary to compute the partial derivatives of the global cost $\bar{J}(k)$ with respect to all the parameters to be optimized (see (23), (21), (22)) as explained in Section III, are reported. We remind that the cost to be minimized is $\bar{J}(k) = \bar{J}_D(k) + \bar{J}_P(k)$, where $\bar{J}_D$ is the distortion term $\bar{J}_D(k) = \underset{\{x,v\}}{E}\left\{[x-u_2]^2\right\} = E\{[x - \hat{\gamma}_2(\mathbf{y}_2, \mathbf{w}_2(k))]^2\}$ and $\bar{J}_P$ the penalty term $\bar{J}_P(k) = \underset{\{x,v\}}{E}\left\{\kappa_P f(\bar{P}(k))\right\} = E\{\kappa_P \left[\max\left(\bar{P}(k)-1,0\right)\right]^2\}$. At iteration step k, the average transmission power is computed by Eq. 18 after the generation of a suitable number N of different realizations of the stochastic variables $\left\{x^i(k), v^i(k)\right\}_{i=0}^{N-1}$. We also remind that

$$\begin{aligned}
x^i(k) =& y_1^i(k),\; u_{11}^i(k) = \sigma_B\left[\alpha_B \hat{\gamma}_{11}(y_1^i(k), \mathbf{w}_{11}(k))\right],\\
u_{12}^i(k) =& u_{11}^i(k)\hat{\gamma}_{12}^i(y_1^i(k), \mathbf{w}_{12}(k)),\\
y_{21}^i(k) =& u_{11}^i(k),\; y_{22}^i(k) = u_{12}^i(k) + v^i(k),\\
u_2^i(k) =& \hat{\gamma}_2(y_{21}^i(k), y_{22}^i(k), \mathbf{w}_2(k)).
\end{aligned}$$

The partial derivatives of J_D with respect to the signals are listed hereinafter.

$$\frac{\partial \bar{J}_D(k)}{\partial u_2(k)} = \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial J_D(k)}{\partial u_2^i(k)} = -\frac{2}{N}\sum_{i=1}^{N}\left[x^i(k) - u_2^i(k)\right], \tag{28}$$

$$\begin{aligned}
\frac{\partial \bar{J}_D(k)}{\partial u_{11}(k)} =& \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial J_D(k)}{\partial u_{11}^i(k)} = \\
=& \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial J_D(k)}{\partial u_2^i(k)}\left[\frac{\partial u_2^i(k)}{\partial y_{21}^i(k)} + \frac{\partial u_2^i(k)}{\partial y_{22}^i(k)}\hat{\gamma}_{12}(y_1^i(k), \mathbf{w}_{12}(k))\right],
\end{aligned}\tag{29}$$

$$\frac{\partial \bar{J}_D(k)}{\partial u_{12}(k)} = \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial J_D(k)}{\partial u_2^i(k)}\frac{\partial u_2^i(k)}{\partial y_{22}^i(k)}u_{11}^i(k). \tag{30}$$

The partial derivatives of J_P with respect to the signals are listed hereinafter.

$$\frac{\partial \bar{J}_P(k)}{\partial u_2(k)} = \frac{\kappa_P}{N}\sum_{i=0}^{N-1}\frac{\partial f(\bar{P}(k))}{\partial u_2^i(k)} = 0, \tag{31}$$

$$\frac{\partial \bar{J}_P(k)}{\partial u_{12}(k)} = \frac{1}{N}\sum_{i=0}^{N-1}\frac{\partial \bar{J}_P(k)}{\partial u_{12}^i(k)} = \frac{2\kappa_P}{N}\frac{\partial f(\bar{P}(k))}{\partial \bar{P}(k)}\sum_{i=0}^{N-1}u_{12}^i(k), \tag{32}$$

$$\frac{\partial \bar{J}_P(k)}{\partial u_{11}(k)} = \frac{2\kappa_P}{N}\frac{\partial f(\bar{P}(k))}{\partial \bar{P}(k)}\sum_{i=0}^{N-1}u_{11}^i(k)\hat{\gamma}_{12}^2(y_1^i(k), \mathbf{w}_{12}(k)). \tag{33}$$

After the computation of the derivatives, the values of the three neural networks' weights, $\mathbf{w}_{11}(k), \mathbf{w}_{12}(k), \mathbf{w}_2(k)$, are updated through the stochastic technique described by Eq. 19. For further details on the implementation of the algorithm, see [17].

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