# Control of Nonholonomic Mobile Robots Based on the Transverse Function Approach

Pascal Morin and Claude Samson
INRIA
2004, Route des Lucioles
06902 Sophia-Antipolis Cedex, France

***Abstract*—The problem of stabilizing reference trajectories – also referred to as the trajectory tracking problem– for nonholonomic mobile robots is revisited. Theoretical difficulties and impossibilities which set inevitable limits to what is achievable with feedback control are surveyed, and properties of kinematic control models are recalled, with a focus on controllable driftless systems which are invariant on a Lie group. This geometric framework takes advantage of ubiquitous symmetry properties involved in the motion of mechanical bodies. The Transverse Function approach, a control design method developed by the authors for the past few years, is reviewed. A salient feature of this approach, which singles it out of the abundant literature devoted to the subject, is the obtention of feedback laws which unconditionally achieve the practical stabilization of arbitrary reference trajectories, including fixed points and non-admissible trajectories. This property is complemented with novel results showing how the more common property of asymptotic stabilization of a large class of admissible trajectories can also be granted with this type of control. Application to unicycle-type and car-like vehicles is presented and illustrated via simulations. Complementary issues (transient maneuvers monitoring, extensions of the approach to systems which are not invariant on a Lie group, ...) are also addressed with the concern of practicality.**

***Index Terms*—wheeled robot, nonholonomic system, unicycle, car, stabilization, trajectory tracking, Lie group, transverse function.**

## I. INTRODUCTION

Nonholonomic systems, ranging from unicycle and car-like vehicles, possibly equipped with trailers, to more original systems like rolling spheres [3], [11], [15], snake-like robots [12], [13], snakeboards and roller-racers [16], [18], etc., abound in Robotics. All these mechanical systems share strong controllability properties, but the nonholonomic kinematic constraints which characterize their motion render the associated control design problem quite challenging, as illustrated by Brockett's theorem [4] proving the non-existence of pure-state feedbacks for the asymptotic stabilization of fixed configurations. This difficulty has had the effect of partitioning the research on nonholonomic systems feedback control into two distinct sub-problems, namely i) fixed point asymptotic stabilization relying on highly nonlinear techniques, and ii) asymptotic stabilization of other feasible trajectories based on more classical linear and nonlinear techniques –see, e.g., [27] for more details and references. Within the stream of papers devoted to these problems, [9] addressed the control of a unicycle-type vehicle in a different way which attracted our attention and inspired the development of the Transverse Function (TF) approach at the core of the present paper. The focus on the aforementioned sub-problems has produced solutions which apply to many practical situations. However, it matters to realize that this research activity, undertaken during more than a decade, has not exhausted the subject. First, concerning the asymptotic stabilization of fixed configurations, all attempts to achieve fast convergence and robust stability have failed –to our knowledge. Then, it has been shown that the more general problem of asymptotic stabilization of feasible trajectories in its full generality is essentially unsolvable. More precisely, an important result by Lizàrraga [20] basically proves that the search for a causal feedback control scheme capable of stabilizing "any" feasible reference trajectory for this type of system is vain. Whatever the chosen control strategy there always exists a feasible trajectory that this control is unable to stabilize asymptotically, eventhough any feasible trajectory taken separately can be asymptotically stabilized. This limitation has no equivalence in Linear Control Theory and is an ever lasting source of frustration that control designers and roboticists have to live with. Finally, the problem of feedback stabilization of non-feasible trajectories has seldom been addressed.

These theoretical obstructions and shortcomings have comforted us with the idea that the control problem for this class of systems should primarily focus on an objective less demanding, and thus more open, than the *asymptotic* stabilization of the origin of some error-system. Such an objective may consist, for instance, in the asymptotic stabilization of a small set containing the origin of the error-system, thus leaving the asymptotic stabilization of the origin itself as a complementary possibility rather than a systematic requirement. This type of objective (small bounded error) is also more in accordance with what can be achieved in practice. For this reason it is common to use the generic denomination of *practical* stabilization when referring to it. The TF approach is a control design method that yields practical stabilizers for nonholonomic systems. It does not (cannot) overcome the aforementioned obstructions, but it goes further than other control methods because it more fully exploits the local controllability property of the systems by providing feedback controllers theoretically capable of stabilizing in a practical manner any trajectory, even non-feasible, with arbitrary tracking precision. The benefits that can be gained from these controllers are numerous for robotic applications, due in particular to the possibility of using the same controller for all reference trajectories, including non-feasible ones. Let us comment some more on this.

First, it follows from Lizàrraga's result that no switching strategy between a finite set of (complementary) feedback controllers, however sophisticated it is, can yield a solution to the problem of stabilizing "any" feasible reference trajectory asymptotically. This is a critical issue when the vehicle must operate in fully autonomous mode and no *a priori* information on the reference motion is available. In this respect, the guarantee of uniformly bounded tracking errors and the possibility of tuning the ultimate tracking precision *independently* of the reference motion is a strong asset of the TF approach.

Second, eventhough non-feasible trajectories cannot, by definition, be *asymptotically* stabilized (the tracking error cannot converge to zero) the property of local controllability of nonholonomic systems implies that any non-feasible trajectory can be approximated with arbitrary good precision by feasible ones. Several algorithms generating *open-loop* control inputs have been proposed to solve this approximation problem [19], [37]. On the other hand, the same problem has seldom been addressed with a feedback control point of view, in spite of the importance of this issue for various robotic applications. For instance, the path planning problem in a cluttered environment can be significantly simplified by the removal of the constraint of feasibility of the planned trajectory. Considering a platooning scenario, the problem of "following" a leading vehicle engaged in maneuvers constitutes another example of the usefulness of the possibility of stabilizing non-feasible trajectories (see e.g. [1], [2]). Finally, the control of nonholonomic mobile manipulators (i.e. a robotic arm mounted on a nonholonomic mobile platform) is much simplified when the platform can track non-feasible trajectories, since the problem is essentially reduced to the one of controlling a holonomic mobile manipulator [10].

Now, it is also important to realize that practical stabilization, which is at the core of the TF approach, is not opposed to the achievement of stronger properties. For instance, one of the objectives of the present study is to show that a proper tuning allows for the asymptotic stabilization of feasible trajectories, thus making these controllers also competitive with classical control laws within their own domain of operation. The theoretical foundations of the TF approach have been published in [23], [24]. Complementary results, some theoretical, others more application oriented, have also been published in control journals or conferences. Although an exhaustive presentation of these results is not possible here, the idea is to provide the reader with enough background material and explanations so that he can successfully implement the approach for robotic applications involving classical systems like unicycle and car-like robots, and also develop new control strategies for other systems. Note that the solution here developed for car-like vehicles has not been published before.

The paper is organized as follows. In Section II some properties of kinematic control models of nonholonomic systems are recalled, with a focus on systems which are invariant under a certain Lie group operation. This class of systems contains several examples of interest (unicycles, chained systems, rolling spheres, etc.) and its rich and generic structure allows for the derivation of results applicable to many other systems. In particular, the TF approach is best exposed in this framework although it also applies to systems which are not invariant on a Lie group. This geometric framework has also been used in various robot control studies [5], [17], [32]. Section III is devoted to the TF approach. After recalling the basics of the approach –as developed in [24]– new results about the asymptotic stabilization of feasible reference trajectories are presented and illustrated on three and four dimensional chained systems. Sections IV and V are devoted to the application of the approach to unicycle-type and car-like vehicles by transposing the results obtained for the corresponding chained systems. In both cases, simulation results illustrate various aspects of the controller's performance and complementary practical issues are addressed.

## II. The geometry of kinematic control models

### A. Recalls on kinematic models

For completeness, and also to introduce the notation used thereafter, basic properties of kinematic models of nonholonomic systems (see e.g. [6], [14]) are first recalled. Kinematic equations of nonholonomic mechanical systems are encompassed by driftless control systems of the form

$$\dot{g} = \sum_{i=1}^{m} X_i(g)u_i \tag{1}$$

with $g$ belonging to a $n$-dimensional manifold $G$, $X_1, \ldots, X_m$ the system's control vector fields (v.f.) representing feasible directions of instantaneous motion compatible with the nonholonomic constraints, and $u = (u_1, \ldots, u_m)'$ the control vector, with $z'$ denoting the transpose of a vector $z$. The system's nonholonomy is characterized by the fact that $m < n = \dim(g)$. The kinematic model of a mechanical system is not unique. It depends on the choice of the state $g$ used to represent the system's configuration and the way $\dot{g}$ is decomposed along $m$ independent directions. For example, a standard model for unicycle-type vehicles is

$$\left\{\begin{array}{rcl} \dot{x} & = & u_1 \cos\theta \\ \dot{y} & = & u_1 \sin\theta \\ \dot{\theta} & = & u_2 \end{array}\right. \tag{2}$$

but it is well known that the 3D chained system can also be used as a local model. Recall that the equations of the $n$D chained system with two control inputs are

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ &\vdots \\ \dot{x}_n &= u_1 x_{n-1} \end{cases} \tag{3}$$

Similarly, car-like vehicles can be modeled either by the equations

$$\begin{cases} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_1 (\tan\varphi)/L \\ \dot{\varphi} &= u_2 \end{cases} \tag{4}$$

with $\varphi \in (-\pi/2, \pi/2)$ denoting the steering angle and $L$ the distance between the rear and front wheels' axles, or by the 4D chained system.

Systems (2), (3), and (4) are particular cases of the general system (1), with $m = 2$. In addition, they are controllable at any point, i.e. the set of points reachable from any point during an arbitrary (non-zero) amount of time by using bounded controls contains a neighborhood of this point. For a driftless system (1) with smooth v.f., local controllability at $g$ is granted by[1] the satisfaction at $g$ of the so-called Lie Algebra Rank Condition (LARC) involving iterated Lie brackets of the system v.f. [7], [33]. This condition requires that one can find $n$ independent vectors in the set

$$\{X_i(g), [X_i, X_j](g), [X_i, [X_j, X_k]](g), \dots\}$$

with $i, j, k, \dots \in \{1, \dots, m\}$, and the Lie bracket $[X, Y]$ of two v.f. $X$ and $Y$ defined (in coordinates $x$) by $[X, Y](x) = \frac{\partial Y}{\partial x}(x)X(x) - \frac{\partial X}{\partial x}(x)Y(x)$. For instance, for the 3D chained system the vectors $X_1(x) = (1, 0, x_2)'$, $X_2(x) = (0, 1, 0)'$, and $X_3(x) = [X_1, X_2](x) = (0, 0, -1)'$ form a basis of $\mathbb{R}^3$ for any $x$. To avoid non-essential technicalities the following assumptions are made throughout the paper. They are satisfied by Systems (2), (3), and (4).

**Assumption 1** *For System* (1)

1) *The state space $G$ is a connected manifold,*
2) *The control v.f. $X_1, \dots, X_m$ are independent over $\mathbb{R}$, i.e. $(\sum_{i=1}^m \lambda_i X_i(g) = 0 \ \forall g) \Longrightarrow \lambda_1 = \dots = \lambda_m = 0$, with the $\lambda_i$'s denoting constant scalars,*
3) *The LARC is satisfied at any $g$.*

### B. Systems on Lie groups

*1) Definition and examples:* An important structural property of Systems (2) and (3) is that their v.f. are *left-invariant* with respect to a Lie group operation. Recall (see e.g. [39]) that a Lie group $G$ is a smooth manifold endowed with a "smooth" group law $(g_1, g_2) \mapsto g_1 g_2$, i.e. *i)* the mapping is associative, *ii)* there exists an element $e$ (the unit element) such that $ge = eg = g$ for all $g$, *iii)* for any $g$, there exists an element $g^{-1}$ (the inverse of $g$) such that $gg^{-1} = g^{-1}g = e$, *iv)* the mapping $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is smooth. A v.f. $X$ defined on a Lie group $G$ is left-invariant if

$$\forall g_1, g_2 \in G : \quad dL_{g_1}(g_2).X(g_2) = X(g_1 g_2)$$

with $L_{g_1}$ the *left translation* by $g_1$, defined by $L_{g_1}(g_2) = g_1 g_2$, and $df(p)$ the differential of a mapping $f$ at a point $p$. The set of left-invariant v.f., often denoted as $\mathfrak{g}$, is called the Lie algebra of the group. It is a vector space of the same dimension (over $\mathbb{R}$) as the group. Then we say that (1) is a system on a Lie group if the associated state space $G$ is a Lie group and each control v.f. $X_i$ is left-invariant. An equivalent definition in term of trajectories, probably more intuitive, is that, given any control input $u(t)$ ($t \in [0, T]$), any solution to the system can be deduced from another solution via a left translation by a constant element. More precisely, if $g_1(t)$ and $g_2(t)$ denote two solutions to (1) then $g_2(t) = g_2(0)g_1(0)^{-1}g_1(t)$, $\forall t \in [0, T]$. This geometric property is shared (at the kinematics level) by all rigid bodies with the associated Lie group $SE(3)$ or one of its sub-groups $SE(2)$, $SO(3)$, etc. (see e.g. [32] for a detailed exposition). For example, (2) is a system on the Lie group $SE(2)$ whose group operation is defined by

$$g_1 g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\theta_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix} \tag{5}$$

with $g_i = (x_i, y_i, \theta_i)$ and $R(\theta)$ the matrix of rotation in the plane of angle $\theta$. The unit element is $e = (0, 0, 0)$ and the inverse of $g = (x, y, \theta)$ is

$$g^{-1} = \begin{pmatrix} -R(-\theta) \begin{pmatrix} x \\ y \end{pmatrix} \\ -\theta \end{pmatrix} \tag{6}$$

System (3) is also a system on a Lie group, with the group product $xy$ of two elements $x, y \in \mathbb{R}^n$ defined by

$$(xy)_i = \begin{cases} x_i + y_i & \text{if } i = 1, 2 \\ x_i + y_i + \sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!} x_j & \text{otherwise} \end{cases} \tag{7}$$

with $(xy)_i$ the $i$-th component of $xy$. Let us recall that this system can be used as a kinematic model of many wheeled robots, like unicycles or cars with trailers [36]. Another well known example in robotics of a system on a Lie group is the rolling sphere, also referred to as the ball-plate system [8], [31], [34]. The associated Lie group is $SO(3) \times \mathbb{R}^2$ with the group law inherited from the group law of $SO(3)$ and the vector addition in $\mathbb{R}^2$ (see e.g. [28] for more details). On the other hand the car model (4) is not a system on a Lie group. This follows from the fact that, contrary to the previously mentioned examples, the dimension over $\mathbb{R}$ of the system's Lie algebra is not equal to the dimension of the state space. Nevertheless, the 4D chained system is a system on a Lie group and it is also used as a kinematic model for car-like vehicles. This contradiction is only apparent because the transformation of System (4) into the 4D chained system

[1] and equivalent to, when the control v.f. are real analytic,

involves a change of control variables on top of a change of state coordinates. Whereas the property of left-invariance is preserved by a change of coordinates, a complementary change of control variables is always needed to transform a non-invariant system into an invariant one. The reader interested in these issues is referred to [29, Sec. 2.2.1] for more details.

Since the Lie algebra $\mathfrak{g}$ of a Lie group $G$ is a $n$-dimensional vector space one can look for $n$ v.f. that define a basis of $\mathfrak{g}$. Given a (left-invariant) System (1) on $G$, one deduces from Assumption 1 that a basis $X = \{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$ is obtained by choosing $X_1, \ldots, X_m$ as the control v.f. of System (1) and $X_{m+1}, \ldots, X_n$ as independent iterated Lie brackets of $X_1, \ldots, X_m$. For example, in the case of the group $SE(2)$ associated with the unicycle a possible basis of $\mathfrak{g}$ is

$$X = \{X_1, X_2, [X_1, X_2]\} \tag{8}$$

In the case of the group associated with the chained system (3) a possible basis is

$$X = \{X_1, X_2, (\mathrm{ad}X_1)(X_2), \ldots, (\mathrm{ad}^{n-2}X_1)(X_2)\} \tag{9}$$

with $(\mathrm{ad}^p X)(Y)$ defined recursively by the relations $(\mathrm{ad}^1 X)(Y) = (\mathrm{ad}X)(Y) = [X, Y]$ and $(\mathrm{ad}^p X)(Y) = [X, (\mathrm{ad}^{p-1}X)(Y)]$ for $p \geq 2$.

*2) Error-system:* To stabilize a reference trajectory $g_r(.)$ for System (1) an *error* between the desired (reference) state and the actual state of the system must be defined in the first place. When the system under consideration is invariant on a Lie group, a "natural" tracking error is $\tilde{g}(t) := g_r(t)^{-1}g(t)$. The problem of stabilizing $g_r$ can then be expressed as the problem of stabilizing the unit element $e$ for the *error-system* whose state is $\tilde{g}$, since $\tilde{g}(t) = e$ is equivalent to $g(t) = g_r(t)$. Let us first assume that $g_r$ is constant over time. Then, by the invariance property one has

$$\dot{\tilde{g}} = dL_{g_r^{-1}}(g)\dot{g} = \sum_{i=1}^{m} X_i(\tilde{g})u_i$$

This is the error-system equation and it is the same as the equation of the initial system, thus justifying the adjective "natural" that we have associated with the error $\tilde{g} = g_r^{-1}g$.

Let $R_g$ denote the right translation operator defined by $R_{g_2}(g_1) := g_1 g_2$ $(= L_{g_1}(g_2))$. When $g_r(t)$ varies with time the above error equation becomes (see Relation (74) in Appendix A):

$$\begin{aligned} \dot{\tilde{g}} &= dL_{g_r^{-1}}(g)\dot{g} + dR_g(g_r^{-1})\tfrac{d}{dt}g_r^{-1} \\ &= \sum_{i=1}^{m} X_i(\tilde{g})u_i + P(\tilde{g}, g_r, \dot{g}_r) \end{aligned} \tag{10}$$

with

$$P(\tilde{g}, g_r, \dot{g}_r) = -dR_{\tilde{g}}(e)dL_{g_r^{-1}}(g_r)\dot{g}_r$$

Now, if $X = \{X_1, X_2, \ldots, X_n\}$ denotes a basis of the group's Lie algebra $\mathfrak{g}$ there exists a vector-valued time function $v_r = (v_{r,1}, \ldots, v_{r,n})'$ such that (omitting the time index) $\dot{g}_r = \sum_{i=1}^{n} X_i(g_r)v_{r,i}$. To further simplify the notation we will write $\dot{g}_r = X(g_r)v_r$. Note that this *notation* coincides, when $G = \mathbb{R}^n$, with the product of the matrix $X(g) = (X_1(g)\ X_2(g)\ \ldots\ X_n(g))$ by the vector $v_r$. Using this decomposition of $\dot{g}_r$ in the expression of $P$ one obtains (see Relation (75) in Appendix A):

$$P(\tilde{g}, g_r, \dot{g}_r) = -dL_{\tilde{g}}(e)\mathrm{Ad}(\tilde{g}^{-1})X(e)v_r \tag{11}$$

with Ad the so-called *adjoint representation* defined by

$$\begin{aligned} \mathrm{Ad}(\sigma) &:= dJ_\sigma(e) \\ &= dL_\sigma(\sigma^{-1})dR_{\sigma^{-1}}(e) = dR_{\sigma^{-1}}(\sigma)dL_\sigma(e) \end{aligned} \tag{12}$$

with $J_\sigma(\tau) = \sigma\tau\sigma^{-1}$. From what precedes a concise way of writing the error-system equation (10) is

$$\dot{\tilde{g}} = X(\tilde{g})(Cu - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \tag{13}$$

with $C = (I_m \,|\, 0_{m\times(n-m)})'$, $I_m$ the $(m \times m)$ identity matrix, and $\mathrm{Ad}^X$ the expression of the Ad operator in the basis $X$, i.e. the (invertible) matrix-valued function defined by $\mathrm{Ad}(\sigma)X(e)v := X(e)\mathrm{Ad}^X(\sigma)v$. This expression is a generalization of the original system's equation (1) which, with the notation introduced above, writes as

$$\dot{g} = X(g)Cu \tag{14}$$

*3) Linearized equations:* Given a control system $\dot{\xi} = f(\xi, u)$ on $\mathbb{R}^n$, admitting $(\xi = 0, u = 0)$ as an equilibrium (i.e. $f(0, 0) = 0$), the linear approximation of this system at this equilibrium is $\dot{\xi} = A\xi + Bu$ with $A = \frac{\partial f}{\partial \xi}(0, 0)$ and $B = \frac{\partial f}{\partial u}(0, 0)$. When the linear approximation is controllable, classical linear control design techniques provide linear feedback control laws $u = K\xi$ which exponentially stabilize $\xi = 0$ for the closed-loop system –the problem reduces to calculating a suitable gain matrix $K$ such that $A + BK$ is Hurwitz stable. Moreover, any of these feedbacks also (locally) exponentially stabilizes $\xi = 0$ for the original nonlinear system. This well-known result illustrates the importance of linear control theory for nonlinear systems whose linear approximations are controllable.

As pointed out above two issues systematically arise when attempting to apply linear control techniques to nonlinear systems: *i)* the existence of an equilibrium of interest, and *ii)* the controllability (or at least, the stabilizability) of the linear approximation at this point. Concerning the first one, using the fact that $\mathrm{Ad}(e)$ is the identity operator, Equation (13) tells that $\tilde{g} = e$ is an equilibrium of the error-system only if $v_r$ belongs to the image set of $C$, i.e. $v_r = Cu_r$ with $u_r \in \mathbb{R}^m$. In view of (14) this just means that $(g_r(t), u_r(t))$ must be one of the system's solutions. It is common to say in this case that the reference trajectory is *feasible*, or *admissible*. We will assume at this point that the reference trajectory is feasible so that the error-system equation can be written as

$$\dot{\tilde{g}} = X(\tilde{g})(C\tilde{u} - (\mathrm{Ad}^X(\tilde{g}^{-1}) - I_n)Cu_r) \tag{15}$$

with $\tilde{u} := u - u_r$. The pair $(\tilde{g}, \tilde{u}) = (e, 0)$ is an equilibrium of this system, and the control objective is to stabilize this point.

Let us now examine the question of controllability of the associated linearized system at this point. First, when a control system evolves on an $n$-dimensional manifold $G$ its linearization at an equilibrium point makes sense only after defining coordinates to represent the system's state as a vector in $\mathbb{R}^n$. Local coordinates in the neighborhood of $e$ can be defined in several ways, but the most general methods rely on the *exponential mapping*, $\exp : \mathfrak{g} \longrightarrow G$, which defines a local diffeomorphism from a neighborhood of the origin of $\mathfrak{g}$ to a neighborhood of $e$. Let us recall that given a v.f. $Y \in \mathfrak{g}$, $\exp(Y)$ denotes the value at time $t = 1$ of the solution of $\dot{g} = Y(g)$ with initial condition $g(0) = e$. For example, so-called *coordinates of the first kind*, $\xi$, are defined by the relation $g := \exp(X\xi)$, with $X$ a basis of $\mathfrak{g}$. Let us illustrate this possibility in the case of the 3D chained system which is invariant on the Lie group $\mathbb{R}^3$ endowed with the group operation (7). Since the state manifold is $\mathbb{R}^3$, note that $g = x$ already defines a system of coordinates. Consider the Lie algebra basis defined by (9) (with $n = 3$). It follows from (3) that $X_3 = (0, 0, -1)'$. Then the vector of coordinates $\xi$ of a group's element $x$ is related to the canonical coordinates $x_i$ of $x$ by computing the solution of

$$\begin{cases} \dot{y}_1 = \xi_1 \\ \dot{y}_2 = \xi_2 \\ \dot{y}_3 = \xi_1 y_2 - \xi_3 \end{cases} \quad , \quad y(0) = 0$$

at time $t = 1$ and by setting the result equal to $x$. This yields

$$x = \exp(X\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \frac{\xi_1 \xi_2}{2} - \xi_3 \end{pmatrix} \tag{16}$$

For the Lie group $\mathbb{R}^4$ endowed with the group operation (7) (i.e. the one associated with the 4D chained system) and the basis (9), the following expression of the exp function is obtained:

$$x = \exp(X\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \frac{\xi_1 \xi_2}{2} - \xi_3 \\ \frac{\xi_1^2 \xi_2}{6} - \frac{\xi_1 \xi_3}{2} + \xi_4 \end{pmatrix} \tag{17}$$

For any system on a Lie group with $\mathbb{R}^n$ as the state manifold one can use either canonical coordinates $x$ or coordinates of the first kind $\xi$. In what follows the latter set of coordinates is used due to the general applicability of the relations derived with this representation.

Forthcoming relations involve the adjoint representation ad (recall that $(\text{ad } Y)(Z) = [Y, Z]$). A useful relation between ad and the group's adjoint representation Ad is

$$\frac{d}{dt}_{|t=0} \text{Ad}(\exp(tY))Z(e) = (\text{ad } Y)(Z)(e) \tag{18}$$

In a way similar to the definition of $\text{Ad}^X$ we denote by $\text{ad}^X$ the expression of the ad operator in the basis $X$, i.e. $\forall v_1, v_2 \in \mathbb{R}^n$,

$$X(e)\text{ad}^X(v_1)v_2 = (\text{ad } Xv_1)(Xv_2)(e) = [Xv_1, Xv_2](e)$$

The linear approximation of (15) at the equilibrium $(\tilde{g}, \tilde{u}) = (e, 0)$, in the coordinates $\xi$, is (see e.g. [28])

$$\dot{\tilde{\xi}} = -\text{ad}^X(Cu_r)\tilde{\xi} + C\tilde{u} \tag{19}$$

To calculate the matrix $\text{ad}^X(Cu_r)$ a useful relation is

$$\text{ad}^X(v) = \left((c^j_{k1})v | \dots | (c^j_{kn})v\right) \tag{20}$$

with $(c^j_{kp})$ $(p = 1, \dots, n)$ denoting the matrix whose element at row $j$ and column $k$ is $c^j_{kp}$, one of the *structure constants* of the original nonlinear system relative to the chosen Lie algebra basis $X = \{X_1, \dots, X_n\}$. These constants are themselves defined by the relation $[X_k, X_p] = \sum_{j=1}^n X_j c^j_{kp}$. In the case of the $n$D chained system, using the fact that $X_{i+1} = [X_1, X_i]$ and that $[X_j, X_k] = 0$ when neither $j$ nor $k$ is equal to 1, one has

$$c^r_{pq} = \begin{cases} 1 & \text{if } p = 1,\ q \neq 1,\ r = q + 1 \\ -1 & \text{if } q = 1,\ p \neq 1,\ r = p + 1 \\ 0 & \text{otherwise} \end{cases}$$

so that, from (20),

$$\text{ad}^X(Cu_r) = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 \\ -u_{r,2} & u_{r,1} & 0 & 0 & \dots & 0 \\ 0 & 0 & u_{r,1} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & u_{r,1} & 0 \end{pmatrix} \tag{21}$$

Let us mention two important properties of Eq. (19). First, this equation is completely general for systems on a Lie group. Then, the associated state and control matrix can be computed without determining the coordinates $\xi$ explicitly. This is exploited in [28] to derive a necessary condition for the controllability of System (19) in the case of a constant reference input $u_r$, by inspection of the control Lie algebra structure only.

From (19), when $u_r$ is constant, the linearized error system is stabilizable iff the pair $(\text{ad}^X(Cu_r), C)$ is stabilizable. In the case of chained systems, and in view of (21), this condition is equivalent to $(u_{r,1}, u_{r,2}) \neq (0, 0)$ when $n = 3$, and $u_{r,1} \neq 0$ when $n > 3$. These conditions upon $u_r$ may be interpreted as *persistence* conditions which, if they are satisfied, ensure the stabilizability of the linearized error system and, subsequently, the existence of exponential stabilizers whose expression can be obtained either by applying classical linear control design techniques or via slightly more advanced nonlinear control techniques yielding a larger domain of stability under slightly weaker persistence conditions (for instance, $\forall t : \int_t^{t+T} u_{r,1}(s)^2 ds > \varepsilon$ for some $T, \varepsilon > 0$). However, a shortcoming of these "classical" linear and nonlinear feedback laws is that they fail to asymptotically stabilize fixed points (for which $u_r = 0$). Nor do they usually give good results when the reference trajectory is not feasible. For instance, the boundedness of the tracking errors may not be ensured.

## III. The Transverse Function control approach

Unless specified otherwise, we assume from now on that the system to be controlled is of the form (1) and on a Lie group so that all relations derived for these systems apply.

The Transverse Function (TF) control approach [24] provides feedback controls which ensure uniform practical stabilization of any reference trajectory, whether this trajectory is feasible or not, whether it is feasible and persistent or reduced to a fixed point. Moreover, we will see that this type of feedback can also yield asymptotic stabilization in cases when classical control techniques allow for this type of stabilization, i.e. when the reference trajectory is persistent. The remainder of the paper is devoted to this approach and its application/particularization to unicycle-type and car-like vehicles.

### A. Basics of the Transverse Function approach

Let:

- $G$ denote the Lie group on which the system's state evolves,
- $X = \{X^1, X^2\}$, with $X^1 = \{X_1, \ldots, X_m\}$ and $X^2 = \{X_{m+1}, \ldots, X_n\}$, denote a basis of the associated Lie algebra $\mathfrak{g}$,
- dist$(.,.)$ denote a left-invariant distance on $G$, i.e. such that $\forall g_{1,2,3} \in G \; : \; \text{dist}(g_1 g_2, g_1 g_3) = \text{dist}(g_2, g_3)$,
- $f$ denote a differentiable function from $\mathbb{T}^{n-m}$, the torus of dimension $(n-m)$, to a neighborhood $\mathcal{U} \in G$ of the group's unit element $e$,
- $\alpha(t) = (\alpha_{m+1}(t), \ldots, \alpha_n(t))'$ denote a smooth curve on $\mathbb{T}^{n-m}$.

The decomposition of $\dot{f}$ on the basis $X$ yields the existence of a matrix-valued function $A$ such that, $\forall (\alpha, \dot{\alpha})$ :

$$\begin{aligned} \dot{f}(\alpha) &= X(f(\alpha))A(\alpha)\dot{\alpha} \\ &= X^1(f(\alpha))A^1(\alpha)\dot{\alpha} + X^2(f(\alpha))A^2(\alpha)\dot{\alpha} \end{aligned} \tag{22}$$

with $A^1(\alpha), A^2(\alpha)$ matrices corresponding to a row decomposition of $A(\alpha)$, i.e.,

$$A(\alpha) = \begin{pmatrix} A^1(\alpha) \\ A^2(\alpha) \end{pmatrix}$$

Define the "modified" tracking error

$$z := \tilde{g} f(\alpha)^{-1} \tag{23}$$

and note that if $f(\alpha)$ is uniformly close to $e$ then $z$ is uniformly close to $\tilde{g}$, since $\text{dist}(\tilde{g}, z) = \text{dist}(z^{-1}\tilde{g}, z^{-1}z) = \text{dist}(f(\alpha), e)$. Note also that $z = e$ implies that $\tilde{g} = f(\alpha)$. Thus, it suffices to have $z$ converge to $e$ in order to have $\tilde{g}$ come close to $e$. Monitoring the tracking error $\tilde{g}$ via the control of $z$ is the central idea of the Transverse Function approach whose name comes from the specific properties of the function $f$ which make the asymptotic stabilization of $z = e$ a simple control problem. More precisely, by using (13) and Relation (77) in Appendix A, one obtains

$$\dot{z} = X(z)\text{Ad}^X(f(\alpha))(\bar{C}(\alpha)\bar{u} - \text{Ad}^X(\tilde{g}^{-1})v_r) \tag{24}$$

with

$$\begin{aligned} \bar{C}(\alpha) &:= (C \mid -A(\alpha)) \\ &= \begin{pmatrix} I_m & -A^1(\alpha) \\ 0 & -A^2(\alpha) \end{pmatrix} \end{aligned} \tag{25}$$

and $\bar{u}' := (u', \dot{\alpha}') = (u_1, \ldots, u_m, \dot{\alpha}_{m+1}, \ldots, \dot{\alpha}_n)$, which may be seen as an *augmented* $n$-dimensional control vector composed of the original $m$ control inputs and the $n-m$ time-derivative components of $\alpha$. Then, if $\bar{C}(\alpha)$ is invertible for any $\alpha$, the feedback

$$\bar{u} = \bar{C}(\alpha)^{-1}\left(\text{Ad}^X(\tilde{g}^{-1})v_r + \text{Ad}^X(f(\alpha)^{-1})\bar{v}\right) \tag{26}$$

transforms the equation of evolution of $z$ into the system

$$\dot{z} = X(z)\bar{v} \tag{27}$$

Therefore, any asymptotic stabilizer $\bar{v}(z)$ of $z = e$ for this system yields a feedback $\bar{u}(g, g_r, u_r, \alpha)$ which makes the tracking error $\tilde{g}$ converge to the image set of the function $f$. The design of such a stabilizer is not difficult because, in view of (27), the variations of $z$ along each of the $n$ possible directions –given by $X_i$, $i \in \{1, \ldots, n\}$– are directly monitored via an independent control input. For example, in the case of the $n$D chained system with the basis $X$ defined by (9), $\bar{v}(z) = (-k_1 z_1, -k_2 z_2, k_3 z_3, \ldots, (-1)^{i-1} k_i z_i, \ldots)'$, with $k_{1,\ldots,n} > 0$, is a global exponential stabilizer of $z = e = 0$. When $\bar{v}(z)$ is an exponential stabilizer of $z = e$ then, along any solution to the controlled system, $\text{dist}(z(t), e)$ and $|\bar{v}(z(t))|$ converge to zero exponentially. In the particular case where the reference trajectory reduces to a fixed point, i.e. when $u_r = 0$, and in view of (26), all components of the extended control $\bar{u}$ also converge to zero exponentially. This in turn implies that the extended state $(\tilde{g}, \alpha)$ converges exponentially to some fixed point $(f(\alpha^{lim}), \alpha^{lim}) \in G \times \mathbb{T}^{n-m}$.

### B. Existence and calculation of transverse functions

In order to apply the control law (26) the matrix $\bar{C}(\alpha)$ must be invertible for every $\alpha \in \mathbb{T}^{n-m}$. From the expression (25) of $\bar{C}$ this property is itself equivalent to the invertibility of $A^2(\alpha)$ for every $\alpha$. The TF theorem given in [24] asserts the equivalence between the existence of functions $f$ which satisfy this property (of transversality w.r.t. the v.f. $X_1, \ldots, X_m$) and the satisfaction of the LARC by $X_1, \ldots, X_m$, i.e. the controllability of the corresponding driftless system. This theorem also provides a general expression for a family of such functions, the usage of which for the 3D and 4D chained systems is detailed next.

In the case of the 3D chained system a possible choice is

$$\begin{aligned} f(\alpha) &= \exp(\varepsilon_1 \sin(\alpha)X_1 + \varepsilon_2 \cos(\alpha)X_2) \\ &= \begin{pmatrix} \varepsilon_1 \sin(\alpha) \\ \varepsilon_2 \cos(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{4} \sin(2\alpha) \end{pmatrix} \end{aligned} \tag{28}$$

with $\varepsilon_1$ and $\varepsilon_2$ any non-zero real numbers. Note that the second equality in (28) can be deduced from (16) by setting

$\xi_1 = \varepsilon_1 \sin(\alpha), \xi_2 = \varepsilon_2 \cos(\alpha)$, and $\xi_3 = 0$. It is simple to check that this function is transverse to the v.f. $X_1$ and $X_2$ of the 3-dimensional chained system. Indeed, one has

$$\dot{f}(\alpha) = \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{2} \cos(2\alpha) \end{pmatrix} \dot{\alpha} = X(f(\alpha)) \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{2} \end{pmatrix} \dot{\alpha}$$

so that, in this case,

$$A^1(\alpha) = \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \end{pmatrix} \quad , \quad A^2(\alpha) = \frac{\varepsilon_1 \varepsilon_2}{2} \tag{29}$$

Note that the Euclidean distance (which is equivalent to a left-invariant distance near the group's unit element) between $f(\alpha)$ and $e = 0$ can be kept as small as desired by choosing $|\varepsilon_1|$ and $|\varepsilon_2|$ small enough.

In the case of the 4D chained system a TF is obtained as the (group) product of two functions:

$$f(\alpha) = f_4(\alpha_4) f_3(\alpha_3)$$

with $\alpha = (\alpha_3, \alpha_4)' \in \mathbb{T}^2$ and

$$\begin{aligned} f_3(\alpha_3) &= \exp(\varepsilon_{31} s\alpha_3 X_1 + \varepsilon_{32} c\alpha_3 X_2) \\ f_4(\alpha_4) &= \exp(\varepsilon_{41} s\alpha_4 X_1 + \varepsilon_{42} c\alpha_4 X_3) \end{aligned}$$

In the above relations we have used the concise notation $s\alpha = \sin(\alpha)$ and $c\alpha = \cos(\alpha)$. Using (7) and (17), this yields:

$$f(\alpha) = \begin{pmatrix} \varepsilon_{41} s\alpha_4 \\ 0 \\ -\varepsilon_{42} c\alpha_4 \\ -\frac{\varepsilon_{41}\varepsilon_{42}}{4} s2\alpha_4 \end{pmatrix} \begin{pmatrix} \varepsilon_{31} s\alpha_3 \\ \varepsilon_{32} c\alpha_3 \\ \frac{\varepsilon_{31}\varepsilon_{32}}{4} s2\alpha_3 \\ \frac{\varepsilon_{31}^2 \varepsilon_{32}}{6} (s\alpha_3)^2 c\alpha_3 \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_{31} s\alpha_3 + \varepsilon_{41} s\alpha_4 \\ \varepsilon_{32} c\alpha_3 \\ \frac{\varepsilon_{31}\varepsilon_{32}}{4} s2\alpha_3 - \varepsilon_{42} c\alpha_4 \\ \frac{\varepsilon_{31}^2 \varepsilon_{32}}{6} (s\alpha_3)^2 c\alpha_3 - \frac{\varepsilon_{41}\varepsilon_{42}}{4} s2\alpha_4 - \varepsilon_{31}\varepsilon_{42} s\alpha_3 c\alpha_4 \end{pmatrix} \tag{30}$$

We leave to the interested reader the task of verifying that, in this case,

$$A^1(\alpha) = \begin{pmatrix} \varepsilon_{31} c\alpha_3 & \varepsilon_{41} c\alpha_4 \\ -\varepsilon_{32} s\alpha_3 & 0 \end{pmatrix}$$

$$A^2(\alpha) = \begin{pmatrix} \frac{\varepsilon_{31}\varepsilon_{32}}{2} & -\varepsilon_{42} s\alpha_4 + \varepsilon_{32}\varepsilon_{41} c\alpha_3 c\alpha_4 \\ -\frac{\varepsilon_{31}^2 \varepsilon_{32}}{6} s\alpha_3 & \begin{matrix} \frac{\varepsilon_{41}\varepsilon_{42}}{2} + \varepsilon_{31}\varepsilon_{42} s\alpha_3 s\alpha_4 \\ -\frac{\varepsilon_{31}\varepsilon_{32}\varepsilon_{41}}{2} s\alpha_3 c\alpha_3 c\alpha_4 \end{matrix} \end{pmatrix} \tag{31}$$

and that sufficient conditions for the invertibility of $A^2(\alpha)$ are

$$|\varepsilon_{41}| > \frac{4}{3} |\varepsilon_{31}| > 0 \quad , \quad |\varepsilon_{42}| > \frac{|\varepsilon_{32}|}{2(\frac{3}{|\varepsilon_{31}|} - \frac{4}{|\varepsilon_{41}|})} > 0 \tag{32}$$

### *C. Transformation of a controllable nonholonomic system into an omnidirectional companion system*

It is conceptually useful to view the TF control approach as a means to transform an initial controllable (left-invariant) system $\dot{g} = X(g)Cu$ into a *companion system* whose state is $\bar{g} := gf^{-1}$ and whose equation of evolution, obtained for instance by setting $g_r = e$ and $u_r = 0$ in (24), is

$$\dot{\bar{g}} = X(\bar{g}) w \tag{33}$$

with $w = \mathrm{Ad}^X(f)\bar{C}\bar{u}$. Since $\dim(w) = \dim(\bar{u})$, and since both matrices $\mathrm{Ad}^X(f)$ and $\bar{C}$ are invertible (provided that $f$ is a TF), this equation indicates that the companion state can be directly modified along any direction of the tangent space (omnidirectionality). Therefore, the companion system is much more easily controlled than the original system. Moreover, thanks to the associativity of the group product, the modified tracking error $z = \tilde{g} f^{-1}$ may also be viewed as the tracking error $z = g_r^{-1}\bar{g}$ associated with the companion system. The corresponding equation, given by (24), should then be written as follows

$$\dot{z} = X(z)(w - \mathrm{Ad}^X(z^{-1})v_r)$$

### *D. Transverse function shaping for the asymptotic stabilization of feasible trajectories*

Throughout this section it is assumed that the reference trajectory $g_r$ is feasible, i.e. $v_r = Cu_r$. When $\bar{v}(z)$ is an exponential stabilizer of $z = e$ for System (27) the tracking error $\tilde{g}(t)$ converges to the set $f(\mathbb{T}^{n-m})$ contained in a neighborhood of $e$. This convergence property is clearly a desirable feature, but in many cases one would like to guarantee the convergence of $\tilde{g}(t)$ to $e$. This is possible only if there exists $\alpha \in \mathbb{T}^{n-m}$ such that $f(\alpha) = e$. One easily verifies that the latter equality cannot be satisfied in the case of the TFs (28) and (30) proposed previously. This in turn raises the question of the construction of TFs which admit $e$ as an image point, and also, more generally, of criteria for the selection of an adequate function amidst all possibilities. In the case of the functions (28) and (30) another matter related to this issue is the choice of the parameters $\varepsilon_i$ $(i = 1, 2)$ and $\varepsilon_{ij}$ $(i = 3, 4\ , j = 1, 2)$, knowing that large values for these parameters increase the maximal distance between $f(\alpha)$ and $e$, whereas small values render $\bar{C}(\alpha)$ close to singular, yielding large control gains and problems commonly associated with such gains. Note also that nothing forbids the use of time-varying parameters, provided that the property of transversality is preserved all the time. The design of TFs is still a largely open research domain and, in what follows, the present paper only explores the connection existing between the choice of a TF and the possibility of achieving asymptotic stabilization in the case of persistent feasible trajectories, as a complement to the practical stabilization objective which, as explained above, is achieved whatever the chosen TF and whatever the reference trajectory.

We define a *generalized transverse function* as a smooth function $\bar{f} : (\alpha, \alpha_r) \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m} \mapsto \bar{f}(\alpha, \alpha_r) \in G$ such that

1) $\bar{f}$ is *transversal to* $X^1$ *w.r.t.* $\alpha$, i.e. the matrix $A^2(\alpha, \alpha_r)$ defined by the relation $\dot{\bar{f}}(\alpha, \alpha_r) =$

$X^1(\bar{f}(\alpha,\alpha_r))A^1(\alpha,\alpha_r)\dot{\alpha}+X^2(\bar{f}(\alpha,\alpha_r))A^2(\alpha,\alpha_r)\dot{\alpha}$, with $\alpha$ an arbitrary smooth curve and $\alpha_r$ constant, is invertible $\forall(\alpha,\alpha_r)$,

2) $\bar{f}(\alpha_r,\alpha_r)=e$, $\forall\alpha_r\in\mathbb{T}^{n-m}$.

In other words a generalized TF is a function which, besides the variables needed for the satisfaction of the transversality property, depends on as many additional variables which, when equal to the first variables, "shrink" the image of this function to the unit element $e$. This feature may be thought of as a *phase synchronization* property.

Given any TF $f$, it is not difficult to obtain a generalized TF. An example is the function $\bar{f}$ defined by

$$\bar{f}(\alpha,\alpha_r):=f(\alpha_r)^{-1}f(\alpha) \tag{34}$$

The conservation of the transversality property w.r.t. $\alpha$ comes from that, for any smooth curve $\alpha(.)$ and any constant $\alpha_r$,

$$\begin{aligned}\dot{\bar{f}}(\alpha,\alpha_r)&=dL_{f(\alpha_r)^{-1}}(f(\alpha))\dot{f}(\alpha)\\&=dL_{f(\alpha_r)^{-1}}(f(\alpha))X(f(\alpha))A(\alpha)\dot{\alpha}\\&=X(\bar{f}(\alpha,\alpha_r))A(\alpha)\dot{\alpha}\end{aligned}$$

whereas the fact that $\bar{f}(\alpha_r,\alpha_r)=e$ is just a consequence of the definition of the inverse of an element of $G$. In [25] other generalized TFs are proposed to achieve the asymptotic stabilization of fixed equilibrium points for the $n$D chained system. When using such a function in the control law, the convergence of $\tilde{g}$ to $e$ is then obtained when $\alpha$ converges to $\alpha_r$. Are there "good" values of $\alpha_r$ for which this latter convergence can take place when tracking a feasible trajectory? This question is treated next.

From now on $\alpha_r$ is assumed to be constant and the dependence of $\bar{f}$ upon $\alpha_r$ is omitted for the sake of lightening the notation. Let us assume that the feedback control (26) is applied to the system with the TF (34) and with $\bar{v}(z)$ an exponential stabilizer of $e$ for the system (27). Then $z=\tilde{g}\bar{f}(\alpha)^{-1}$ converges exponentially to $e$. The extinction of the transient phase of convergence of $z$ to $e$, characterized by the equality $\tilde{g}=\bar{f}(\alpha)$, leaves us with a differential system in the variable $\alpha$, the so-called *zero dynamics*. If $\alpha=\alpha_r$ is an asymptotically stable equilibrium of this system, then one can prove that $(\tilde{g},\alpha)=(e,\alpha_r)$ is asymptotically stable for the controlled system. Let us thus have a closer look at the system's zero dynamics.

**Proposition 1** *Assume that the reference trajectory is feasible. Then, on the zero dynamics $z=e$ the variable $\bar{\alpha}:=A(\alpha_r)(\alpha-\alpha_r)$ satisfies the equation*

$$P\dot{\bar{\alpha}}=-P\mathrm{ad}^X(Cu_r)\bar{\alpha}+\sum_{i=1}^{r}u_{r,i}\,o_i(\bar{\alpha}) \tag{35}$$

*with $P=(0_{m\times m}|I_{n-m})$ (so that $PC=0$) and $o_i(.)$ denoting a function such that $\lim_{|y|\to 0}\frac{|o_i(y)|}{|y|}=0$.*

The proof is given in [29].

*Remark:* Eq. (35) is related to the linearized equation (19) of the error-system. Indeed, by pre-multiplying both sides of (19) by the matrix $P$ one obtains $P\dot{\tilde{\xi}}=-P\mathrm{ad}^X(Cu_r)\tilde{\xi}$.

Since $\bar{\alpha}$ is a $n$-dimensional vector and $\alpha-\alpha_r$ is only $(n-m)$-dimensional, the components of $\bar{\alpha}$ are not independent. Define $y:=P\bar{\alpha}=A^2(\alpha_r)(\alpha-\alpha_r)$. By the property of transversality $y=0$ if and only if $\alpha=\alpha_r$. By rewriting Eq. (35) as

$$\begin{aligned}\dot{y}&=-P\mathrm{ad}^X(Cu_r)A(\alpha_r)(PA(\alpha_r))^{-1}y++\sum_{i=1}^{r}u_{r,i}\,o_i(y)\\&=-P\mathrm{ad}^X(Cu_r)\begin{pmatrix}A^1(\alpha_r)A^2(\alpha_r)^{-1}\\I_{n-m}\end{pmatrix}y+\sum_{i=1}^{r}u_{r,i}\,o_i(y)\end{aligned} \tag{36}$$

one obtains the following linear approximation of the zero dynamics at the equilibrium $y=0$

$$\dot{y}=-\mathrm{ad}_{21}^X(Cu_r)A^1(\alpha_r)A^2(\alpha_r)^{-1}y-\mathrm{ad}_{22}^X(Cu_r)y \tag{37}$$

where the decomposition of $\mathrm{ad}^X$ in four blocks $\mathrm{ad}_{ij}^X$ $(i,j\in\{1,2\})$ of adequate dimensions has been used. From the above equation this equilibrium is (exponentially) stable iff the feedback control $v=A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ (exponentially) stabilizes the origin of the linear system

$$\dot{y}=-\mathrm{ad}_{22}^X(Cu_r)y-\mathrm{ad}_{21}^X(Cu_r)v \tag{38}$$

Note that the linearized error-system (19) can also be written as

$$\begin{cases}\dot{\xi}_1&=&w\\\dot{\xi}_2&=&-\mathrm{ad}_{22}^X(Cu_r)\xi_2-\mathrm{ad}_{21}^X(Cu_r)\xi_1\end{cases}$$

with $w=\tilde{u}-(ad_{11}^X(Cu_r)\xi_1+ad_{21}^X(Cu_r)\xi_2)$. This is just a dynamic extension of (38) with integrators added at the input control level. When $u_r$ is constant the above linear system does not depend on time and it is well known (and simple to verify) that the stabilizability of the latter system is equivalent to the stabilizability of (38). Therefore, in this case, a necessary condition for the exponential stability of $\alpha-\alpha_r=0$, and subsequently of $\tilde{x}=e$, is the stabilizability of the linearized error-system (19). For the 3D (resp. 4D) chained system we have already seen that this condition is equivalent to $(u_{r,1},u_{r,2})\neq(0,0)$ (resp. $u_{r,1}\neq 0$). This indicates that, as for the problem of asymptotic stabilization of feasible trajectories, the TF approach cannot perform better than classical control methods. But it may perform as well (in the sense of achieving exponential stabilization) and, to this aim, the linear feedback $v=A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ must asymptotically stabilize the origin of (38). For the 3D (resp. 4D) chained system and the TF $\bar{f}(\alpha)=f(\alpha_r)^{-1}f(\alpha)$ with $f$ given by (28) (resp. (30)) we show below that the satisfaction of this condition itself depends on the choice of $\alpha_r$ in relation to the signs of the TF parameters $\varepsilon_i$ (resp. $\varepsilon_{ij}$), $i,j\in\{1,2\}$.

*1) 3D chained system:* From (21), the system (38) specializes to $\dot{y}=\begin{pmatrix}u_{r,2} & -u_{r,1}\end{pmatrix}v$ and, from (29),

$$A^1(\alpha)A^2(\alpha)^{-1}=\begin{pmatrix}\frac{2\cos(\alpha)}{\varepsilon_2}\\-\frac{2\sin(\alpha)}{\varepsilon_1}\end{pmatrix}$$

Therefore, the application of the feedback $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ to this system yields the closed-loop system

$$\dot{y} = 2\left(\frac{u_{r,1}(t)\sin(\alpha_r)}{\varepsilon_1} + \frac{u_{r,2}(t)\cos(\alpha_r)}{\varepsilon_2}\right)y$$

and the following result, with $sign(.)$ denoting the classical sign function and $sign(0)$ chosen equal to either 1 or $-1$ :

**Lemma 1** *($3D$ chained system) Assume that the reference trajectory is feasible with $u_r$ continuous, bounded, and such that $\forall t\ :\ 0 < c \leq |u_{r,1}(t)|$ for some constant c. Then the control (26) with*

*i) $f = \bar{f}$ given by (28), (34) and the following complementary specifications*

$$\left\{\begin{array}{l} \varepsilon_1 = |\varepsilon_1| sign(u_{r,1}(t)), \\ \alpha_r = -\frac{\pi}{2}(= \alpha(0)) \end{array}\right. \tag{39}$$

*ii) $\bar{v}(z)$ an exponential stabilizer of $z = 0$ for the system $\dot{z} = X(z)\bar{v}$ (e.g., $\bar{v}(z) = (-k_1z_1, -k_2z_2, k_3z_3)'$ with $k_{1,2,3} > 0$)*

*locally exponentially stabilizes the zero tracking error $\tilde{g} = 0$.*

This lemma establishes the convergence of the tracking errors to zero under a *persistence* condition upon $u_{r,1}$. Other conditions would result from other possible choices of $\alpha_r$. Note also that the rate of convergence on the zero dynamics is proportional to $|u_{r,1}|$ and to the inverse of $|\varepsilon_1|$.

*2) $4D$ chained system:* In this case, the system (38) specializes to

$$\dot{y} = \begin{pmatrix} 0 & 0 \\ -u_{r,1}(t) & 0 \end{pmatrix} y + \begin{pmatrix} u_{r,2}(t) & -u_{r,1}(t) \\ 0 & 0 \end{pmatrix} v$$

In view of (31), when setting $\alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})'$ one obtains after elementary calculations

$$A^1(\alpha_r)A^2(\alpha_r)^{-1} = \frac{4}{\varepsilon_{31}(\varepsilon_{41} + \frac{4}{3}\varepsilon_{31})} \begin{pmatrix} 0 & 0 \\ (\varepsilon_{31} + \frac{\varepsilon_{41}}{2}) & -1 \end{pmatrix}$$

Therefore, the feedback $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ yields in this case the closed-loop system

$$\dot{y} = u_{r,1}(t) \begin{pmatrix} -\frac{2(2\varepsilon_{31}+\varepsilon_{41})}{\varepsilon_{31}(\frac{4}{3}\varepsilon_{31}+\varepsilon_{41})} & \frac{4}{\varepsilon_{31}(\frac{4}{3}\varepsilon_{31}+\varepsilon_{41})} \\ -1 & 0 \end{pmatrix} y$$

and one deduces the following result :

**Lemma 2** *($4D$ chained system) Assume that the reference trajectory is feasible with $u_r$ continuous, bounded, and such that $\forall t\ :\ 0 < c \leq |u_{r,1}(t)|$ for some constant c. Then the control (26) with*

*i) $f = \bar{f}$ given by (30), (34) and the following complementary specifications*

$$\left\{\begin{array}{l} \varepsilon_{i1} = |\varepsilon_{i1}| sign(u_{r,1}(t))\ , \quad i = 3, 4 \\ \alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})(= \alpha(0)) \end{array}\right. \tag{40}$$

*ii) $\bar{v}(z)$ an exponential stabilizer of $z = 0$ for the system $\dot{z} = X(z)\bar{v}$ (e.g., $\bar{v}(z) = (-k_1z_2, -k_2z_2, k_3z_3, -k_4z_4)'$ with $k_{1,2,3,4} > 0$)*

*locally exponentially stabilizes the zero tracking error $\tilde{g} = 0$.*

As in the case of the 3D chained system other values of $\alpha_r$ also ensure the convergence of $\tilde{g}$ to zero. The important point here was to show that, by a proper choice of the transverse function used in the control law, perfect tracking of feasible reference trajectories can be achieved asymptotically with the complementary insurance of global practical stabilization when the reference trajectory is not feasible, or when it is feasible but the linear approximation of the error-system is not stabilizable. Note that the conditions (39) or (40) upon the parameters $\varepsilon_1$ or $\varepsilon_{i1}$ entering the expression of the TF render this function dependent upon the sign of $u_{r,1}$ and that they introduce discontinuities at the time-instants when this sign changes. Since all previously stated stability results rely on the differentiability of the TF they do not apply *stricto sensu* in this case. For both practical and theoretical reasons discontinuities of the control input at "high frequency" should not be allowed, and one should pre-specify a minimum amount of time $T > 0$ between two successive updates of $\varepsilon_1$ and $\varepsilon_{i1}$ in (39) and (40). Then, one can show that *i)* the control expression remains well-defined $\forall(x, t)$, *ii)* practical stabilization of $\tilde{g} = e$ remains unconditionally granted whatever the reference trajectory, and *iii)* $\text{dist}(\tilde{g}, e)$ continues to be ultimately bounded by a value which can be rendered as small as desired by choosing the absolute values of the TF parameters small enough. The proofs of the last two points much rely on the fact that the distance between two modified tracking errors $z_1 = \tilde{g}f_1(\alpha_1)^{-1}$ and $z_2 = \tilde{g}f_2(\alpha_2)^{-1}$ associated with two different TFs, being equal to the distance between these two functions, is upper-bounded by a value depending only on the size of the parameters entering the expressions of the functions (but not on their signs).

### *E. Tuning of TF parameters*

The values of the TF parameters set ultimate upper-bounds for the tracking errors, *independently* of the reference trajectory. For illustrative purposes let us specify these bounds in the case of the 3D chained system. We first assume that the classical TF (28) is used. Since the tracking error converges to the image set of $f$ one deduces that $|\tilde{g}_1|$, $|\tilde{g}_2|$, and $|\tilde{g}_3|$ are ultimately bounded by $\varepsilon_1, \varepsilon_2$, and $\varepsilon_1\varepsilon_2/4$ respectively. Once $z$ has converged to zero one has $\tilde{g} = f(\alpha)$ and $\bar{v} = 0$ so that, by (26), $\bar{u} = \bar{C}(\alpha)^{-1}\text{Ad}^X(f(\alpha)^{-1})v_r$. This expression allows one to estimate the amplitude of the control inputs involved in the tracking of a given reference trajectory and to relate this amplitude to the TF parameters via the dependence of $\bar{C}(\alpha)^{-1}\text{Ad}^X(f(\alpha)^{-1})$ upon these parameters. The choice of the $\varepsilon_i$'s then becomes a matter of compromise between the tracking precision objective and the requirement of keeping the control inputs within given bounds. Let us now consider the case when the TF is defined according to Lemma 1. The analysis is slightly more involved due to possible sign changes of $\varepsilon_1$. By using the feedback law $\bar{v}(z) = (-k_1 z_1, -k_2 z_2, k_3 z_3)'$ proposed in this lemma one obtains the following ultimate upper-bounds for $|\tilde{g}_1|$, $|\tilde{g}_2|$, and $|\tilde{g}_3|$:

$$2\varepsilon_1 + \frac{4\varepsilon_1}{1-\exp(-k_1 T)}, \varepsilon_2, \frac{\varepsilon_1\varepsilon_2}{4} + \frac{\varepsilon_1\varepsilon_2}{2(1-\exp(-k_3 T))}$$

with $T$ denoting the minimal time interval between two successive updates of $\varepsilon_1$. Recall that these values are only upper-bounds. For example, in the case of feasible reference trajectories satisfying the assumptions of Lemma 1 the property of asymptotic stability ensures null ultimate tracking errors.

## IV. Control of a unicycle-type vehicle

As explained in Section II, the kinematic equations (2) of a unicycle-type vehicle define a (left-invariant) system on the Lie group $SE(2)$, with the group operation specified by (5). From (5) and (6) the tracking error $\tilde{g}$ between $g = (x, y, \theta)'$ and $g_r = (x_r, y_r, \theta_r)'$ is given by

$$\tilde{g} := g_r^{-1} g = \begin{pmatrix} R(-\theta_r)\begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix} \\ \theta - \theta_r \end{pmatrix}$$

Note that the components of this vector are nothing else than the coordinates of the unicycle's situation with respect to the reference frame associated with $g_r$, expressed in the basis of this frame. One also deduces from (5) that

$$dL_g(\bar{g}) = \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}, \; dR_{\bar{g}}(g) = \begin{pmatrix} I_2 & R(\theta)\begin{pmatrix} -\bar{y} \\ \bar{x} \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

Using the above relations, the "perturbation" term $P$ in the error-system equation (10) is defined by

$$P(\tilde{g}, g_r, \dot{g}_r) = -\begin{pmatrix} R(-\theta_r) & \begin{pmatrix} -\tilde{y} \\ \tilde{x} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \dot{g}_r$$

and, from (12),

$$\mathrm{Ad}(g) = \begin{pmatrix} R(\theta) & \begin{pmatrix} y \\ -x \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

With the Lie algebra basis $X$ defined by (8) the matrix-valued function $\mathrm{Ad}^X(.)$ in (13) is defined by

$$\mathrm{Ad}^X(g) = X(e)^{-1}\begin{pmatrix} R(\theta) & \begin{pmatrix} y \\ -x \end{pmatrix} \\ 0 & 1 \end{pmatrix} X(e) \qquad (41)$$

with

$$X(g) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{pmatrix} \qquad (42)$$

### A. Transverse functions

There are many ways to derive TFs. One of them consists in using the general expression given in [24, Th. 1], as we did before for the 3D and 4D chained systems (relations (28) and (30) respectively). In the case of the kinematic model (2) another option consists in using the close kinship between this system and the 3D chained system. Indeed, by setting

$$\begin{cases} \bar{x}_1 = x \\ \bar{x}_2 = \tan\theta \\ \bar{x}_3 = y \end{cases} \quad \text{and} \quad \begin{cases} v_1 = u_1 \cos\theta \\ v_2 = \frac{u_2}{(\cos\theta)^2} \end{cases}$$

System (2) is transformed into the 3D chained system with state $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)'$ and input vector $v = (v_1, v_2)'$. This transformation involves both a change of state coordinates and a change of control inputs, and it is well-defined provided that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Such a transformation is not unique. In fact, there exist more global transformations defined for all angles $\theta \neq \pm\pi$, but this is not important here. Let $\phi$ denote the local diffeomorphism which relates $\bar{x}$ to $g$, i.e. such that $g = \phi(\bar{x}) = (\bar{x}_1, \bar{x}_3, \arctan(\bar{x}_2))'$. Then one can show that the function $f$ defined by $f(\alpha) = \phi(\bar{f}^c(\alpha))$ is transversal to the v.f. $X_1$ and $X_2$ of System (2) provided that $\bar{f}^c$ is transversal to the v.f. of the 3D chained system. For instance, one can take $\bar{f}^c(\alpha) := f^c(\alpha_r)^{-1} f^c(\alpha)$ with $f^c$ the basic TF given by (28) associated with the 3D chained system [2]. This yields:

$$\bar{f}^c(\alpha) = \begin{pmatrix} \varepsilon_1(s\alpha - s\alpha_r) \\ \varepsilon_2(c\alpha - c\alpha_r) \\ \frac{\varepsilon_1\varepsilon_2}{4}(s2\alpha + s2\alpha_r - 4s\alpha c\alpha_r) \end{pmatrix}$$

and

$$f(\alpha) = \begin{pmatrix} \varepsilon_1(s\alpha - s\alpha_r) \\ \frac{\varepsilon_1\varepsilon_2}{4}(s2\alpha + s2\alpha_r - 4s\alpha c\alpha_r) \\ \arctan(\varepsilon_2(c\alpha - c\alpha_r)) \end{pmatrix} \qquad (43)$$

Differentiation w.r.t. $\alpha$ gives:

$$\begin{aligned} \frac{\partial f}{\partial \alpha}(\alpha) &= \begin{pmatrix} \varepsilon_1 c\alpha \\ \varepsilon_1\varepsilon_2(c\alpha(c\alpha - c\alpha_r) - \frac{1}{2}) \\ -\dfrac{\varepsilon_2 s\alpha}{1 + \varepsilon_2^2(c\alpha - c\alpha_r)^2} \end{pmatrix} \\ &= X(f(\alpha))A(\alpha) \end{aligned}$$

with

$$A(\alpha) = \begin{pmatrix} \cos(f_3(\alpha))(\varepsilon_1 c\alpha + \varepsilon_1\varepsilon_2^2(c\alpha(c\alpha - c\alpha_r)^2 - \frac{c\alpha - c\alpha_r}{2})) \\ -\dfrac{\varepsilon_2 s\alpha}{1 + \varepsilon_2^2(c\alpha - c\alpha_r)^2} \\ \frac{\varepsilon_1\varepsilon_2}{2}\cos(f_3(\alpha)) \end{pmatrix}$$

### B. Control

To calculate the feedback control $\bar{u} = (u_1, u_2, \dot{\alpha})'$ specified by (26) there remains to determine *i)* the TF parameters

[2]The group product here involved is the one associated with chained systems as defined by (7).

$\varepsilon_1$, $\varepsilon_2$, and $\alpha_r$, and *ii)* an asymptotic stabilizer $\bar{v}(z)$ of the origin of the system $\dot{z} = X(z)\bar{v}$, with $z := \tilde{g}f(\alpha)^{-1}$. Concerning the first issue the transposition of the study performed for the 3D chained system suggests to choose the TF parameters according to (39) in order to stabilize feasible trajectories asymptotically. As for the second issue a possibility consists in linearizing the closed-loop system (w.r.t. the chosen coordinates) by taking

$$\bar{v}(z) = X(z)^{-1}Kz \tag{44}$$

with $K$ denoting a Hurwitz stable matrix. Indeed, this choice yields the linear closed-loop system $\dot{z} = Kz$ whose origin is exponentially stable. Another possibility, proposed in [2], arises from the concern of limiting the control energy during transient phases corresponding to the convergence of $z$ to $e$. A way to address this issue consists in rewriting the error-system's equation (24) as $\dot{z} = H(z,\alpha)\bar{\bar{u}}$ with

$$\begin{aligned} H(z,\alpha) &= X(z)\mathrm{Ad}^X(f(\alpha))\bar{C}(\alpha) \\ \bar{\bar{u}} &= \bar{u} - \bar{C}(\alpha)^{-1}\mathrm{Ad}^X(\tilde{g}^{-1})v_r \end{aligned}$$

and in determining the control $\bar{\bar{u}}$ which minimizes the cost function $\bar{\bar{u}}'W_1\bar{\bar{u}}$ under the constraint $z'H(z,\alpha)\bar{\bar{u}} + z'W_2z = 0$, with $W_1$ and $W_2$ denoting two symmetric positive definite (s.p.d.) matrices. The underlying idea is to select $W_1$ in order to penalize the physical entries of the control, i.e. the velocities $u_1$ and $u_2$, more than the virtual control input $\dot{\alpha}$. For instance, the fact that $\bar{\bar{u}} = \bar{u} = (u', \dot{\alpha})'$ when $v_r = 0$ suggests to choose $W_1$ diagonal with the first two elements on the diagonal significantly larger than the third one. As for the enforcement of the constraint equality it yields the closed-loop equation $\frac{d}{dt}|z|^2 = -2z'W_2z$ and thus the exponential stabilization of $z = 0$. The solution to this simple constrained minimization problem is:

$$\bar{\bar{u}} = -\frac{z'W_2z}{z'HW_1^{-1}H'z}W_1^{-1}H'z \tag{45}$$

One easily verifies that this is the same as taking:

$$\bar{v}(z) = -\frac{z'W_2z}{z'HW_1^{-1}H'z}\mathrm{Ad}^X(f(\alpha))\bar{C}(\alpha)W_1^{-1}H'z \tag{46}$$

Intuitively, lateral motion of the vehicle can be performed via the execution of either frequent maneuvers involving large and rapidly changing velocity values or less frequent maneuvers involving smaller velocities. Therefore, by penalizing the size of these velocities one can expect to reduce the number of maneuvers during the transient phase of convergence of $z$ to zero. This has been confirmed by many simulations.

### C. Simulation results

For this simulation, the length and width of the unicycle represented on the figures are equal to 2 (meters). A single reference trajectory presenting different properties at different times is used. The time history of the associated reference frame velocity $v_r$ is summarized in the following table.

| $t \in (s)$ | $v_r = (m/s, rad/s, m/s)'$ | properties |
|---|---|---|
| $[0,5)$ | $(0,0,0)'$ | f,npe |
| $[5,10)$ | $(1,0,0)'$ | f,pe |
| $[10,20)$ | $(-1,0,0)'$ | f,pe |
| $[20,25)$ | $(1,0.314,0)'$ | f,pe |
| $[25,30)$ | $(-1,-2\sin(2t),0)'$ | f,pe |
| $[30,35)$ | $(0,0,-1)'$ | nf |
| $[35,40)$ | $(0,0,0)'$ | f,npe |
| $[40,45)$ | $(2,-0.5\sin(3t),0.5)'$ | nf |
| $[45,50)$ | $(0,0,0)'$ | f,npe |

In this table, the abbreviations used to describe the properties of each part of the reference trajectory are:

- *f* and *nf* for feasible and non-feasible respectively, according to whether $v_{r,3}$ is or is not equal to zero;
- *pe* and *npe* for persistent and non-persistent respectively, according to whether $(v_{r,1}, v_{r,2})$ is or is not equal to zero.

The feedback control (26) with $\bar{v}$ defined by (46), which includes a monitoring of the transient phase before the convergence of $z$ to zero, has been used. The parameters chosen for this control are $W_1 = \mathrm{diag}\{1,1,0.01\}$, $W_2 = \mathrm{diag}\{1,1,1\}$. The parameters of the TF are $|\varepsilon_1| = 0.8$, $\varepsilon_2 = 0.5$.

Figure 1 shows the time-evolution of the three components of the modified tracking error $z$. Figures 2-5 attempt to visualize the vehicle's motion in the plane during different phases of the reference trajectory.

The vehicle's real-time motion and the control performance are better visualized by downloading and viewing the corresponding video file *unicycle.avi* contained in a compressed material file of 3.4MB in size available at *http://ieeexplore.ieee.org*.

## V. Control of a car-like vehicle

### A. Kinematic model

The kinematic equations of a car-like vehicle have been recalled in Section II. Unfortunately they do not define a left-invariant system on a Lie group. Nevertheless, we show below that, modulo minor adaptations, the control approach presented in Section III applies to car-like vehicles. As a matter of fact the approach also applies to the more general case of a vehicle with multiple trailers (see [30] for details).

To simplify the notation let us write the kinematic model (4) of a car-like vehicle as

$$\left\{\begin{array}{rcl} \dot{x} &=& u_1\cos\theta \\ \dot{y} &=& u_1\sin\theta \\ \dot{\theta} &=& u_1\eta \\ \dot{\eta} &=& u_\eta \end{array}\right. \tag{47}$$

with $\eta := (\tan\varphi)/L$ and $u_\eta := (1+(\tan\varphi)^2)/L$. This system can also be written as

$$\left\{\begin{array}{rcl} \dot{g} &=& X(g)C(\eta)u_1 \\ \dot{\eta} &=& u_\eta \end{array}\right. \tag{48}$$

with $g = (x,y,\theta)'$ and $X(g)$ (given by (42)) defined as for a unicycle-type vehicle, and $C(\eta) = (1,\eta,0)'$. Note that if

$C$ was a constant vector the above system would be left-invariant on $G = SE(2) \times \mathbb{R}$, with the group law inherited from the group law of $SE(2)$ and the addition on $\mathbb{R}$, i.e.

$$\begin{pmatrix} g_1 \\ \eta_1 \end{pmatrix} \begin{pmatrix} g_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} g_1 g_2 \\ \eta_1 + \eta_2 \end{pmatrix} \tag{49}$$

Let us now consider a reference trajectory $(g_r(t), \eta_r(t))$ for this system and define the tracking error as $(\tilde{g}, \tilde{\eta}) := (g_r^{-1} g, \eta - \eta_r)$. This corresponds to the group product of the inverse of $(g_r, \eta_r)$ by $(g, \eta)$, for the group law (49). One deduces from this definition that (compare with (13)):

$$\left\{ \begin{array}{lcl} \dot{\tilde{g}} & = & X(\tilde{g})(C(\eta)u_1 - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \\ \dot{\tilde{\eta}} & = & \tilde{u}_\eta := u_\eta - \dot{\eta}_r \end{array} \right. \tag{50}$$

with $\mathrm{Ad}^X$ defined by (41) and $\dot{g}_r = X(g_r)v_r$. Following the transverse function approach let us consider a function $f = (f_g, f_\eta) \in SE(2) \times \mathbb{R}$ with the objective of stabilizing the distance between the tracking error $(\tilde{g}, \tilde{\eta})$ and $f$ to zero. For reasons that will become clear later on we consider a function $f$ which depends on both an element $\alpha \in \mathbb{T}^2$ and the independent time-variable $t$, i.e. $f(\alpha, t)$. Define the "modified" tracking error

$$z := \begin{pmatrix} z_g \\ z_\eta \end{pmatrix} := \begin{pmatrix} \tilde{g} f_g^{-1} \\ \tilde{\eta} - f_\eta \end{pmatrix}$$

It follows from relation (77) in the Appendix that

$$\left\{ \begin{array}{l} \dot{z}_g = X(z_g)\mathrm{Ad}^X(f_g(\alpha, t))\big(C(\eta)u_1 - A_\alpha(\alpha, t)\dot{\alpha} \\ \qquad\qquad - A_t(\alpha, t) - \mathrm{Ad}^X(\tilde{g}^{-1})v_r\big) \\ \dot{z}_\eta = \tilde{u}_\eta - \dot{f}_\eta \end{array} \right. \tag{51}$$

with $A_\alpha$ and $A_t$ defined by the relation $\dot{f}_g = X(f_g(\alpha, t))(A_\alpha(\alpha, t)\dot{\alpha} + A_t(\alpha, t))$. Exponential stabilization of $z_\eta$ to zero is simply achieved by setting $\tilde{u}_\eta = \dot{f}_\eta - k_\eta z_\eta$ with $k_\eta > 0$ a control gain. To simplify the exposition we will assume from now on that the convergence of $z_\eta$ to zero has taken place. In doing so we thus neglect transient effects and concentrate on the stabilization of $z_g$ to the origin when $z_\eta = 0$. Since $z_\eta = \eta - \eta_r - f_\eta$ the first equation of (51) then becomes

$$\begin{aligned} \dot{z}_g = X(z_g)\mathrm{Ad}^X(f_g(\alpha, t))\big(&\bar{C}(\alpha, t)\bar{u} \\ &- A_t(\alpha, t) - \mathrm{Ad}^X(\tilde{g}^{-1})v_r\big) \end{aligned} \tag{52}$$

with

$$\bar{C}(\alpha, t) := \big(C(\eta_r(t) + f_\eta(\alpha, t)) \mid -A_\alpha(\alpha, t)\big) \tag{53}$$

and $\bar{u}' := (u_1, \dot{\alpha}')$. If $\bar{C}(\alpha, t)$ is invertible for any $(\alpha, t)$ then the feedback law

$$\begin{aligned} \bar{u} = \bar{C}(\alpha, t)^{-1}\big(&A_t(\alpha, t) + \mathrm{Ad}^X(\tilde{g}^{-1})v_r \\ &+ \mathrm{Ad}^X(f_g(\alpha, t)^{-1})\bar{v}\big) \end{aligned} \tag{54}$$

transforms System (52) into $\dot{z}_g = X(z_g)\bar{v}$. The asymptotic stabilization of $z_g = 0$ via the choice of $\bar{v}(z_g)$ is then a simple matter (see Section V-D).

### B. Transverse functions

Let us now address the design of $f$ in order to ensure the invertibility of the matrix $\bar{C}(\alpha, t)$ for any $(\alpha, t)$. Since $X(z_g)$ is an invertible matrix for any $z_g$ the problem is equivalent to finding $f$ such that the matrix

$$X(f_g)\bar{C}(\alpha, t) = \begin{pmatrix} \begin{matrix} \cos(f_\theta) \\ \sin(f_\theta) \\ \eta_r(t) + f_\eta \end{matrix} & -\dfrac{\partial f_g}{\partial \alpha_1} & -\dfrac{\partial f_g}{\partial \alpha_2} \end{pmatrix}$$

is invertible for any $(\alpha, t)$, with $f_\theta$ the third component of $f_g$. The argument $(\alpha, t)$ of $f_g$, $f_\theta$, and $f_\eta$ is omitted for legibility. An equivalent condition is the invertibility of the matrix

$$H(\alpha, t) = \left( X_{1,\eta_r(t)}(f) \; X_2 \; -\frac{\partial f}{\partial \alpha_1} \; -\frac{\partial f}{\partial \alpha_2} \right) \tag{55}$$

with

$$\begin{array}{c} X_{1,\eta_r}(g, \eta) = (\cos\theta, \sin\theta, \eta_r + \eta, 0)' \\ X_2 = (0, 0, 0, 1)' \end{array} \tag{56}$$

This corresponds to the property of transversality of $f$ w.r.t. the v.f. $X_{1,\eta_r}$ and $X_2$ –compare with the control v.f. of System (47)–, for any value $\eta_r(t)$.

**Lemma 3** *Let $\phi_{\eta_r} : (g, \eta) \longmapsto \bar{x} = \phi_{\eta_r}(g, \eta)$ denote the mapping defined by*

$$\left\{ \begin{array}{l} \bar{x}_1 = x \\ \bar{x}_2 = (\eta + \eta_r(1 - \cos^3\theta))/(\cos^3\theta) \\ \bar{x}_3 = \tan\theta - \eta_r x \\ \bar{x}_4 = y - \eta_r \frac{x^2}{2} \end{array} \right. \tag{57}$$

*with $\eta_r$ an arbitrary constant. Then,*

1) *$\phi_{\eta_r}$ defines a diffeomorphism from $\mathbb{R}^2 \times (-\pi/2, \pi/2) \times \mathbb{R}$ to $\mathbb{R}^4$,*
2) *$\phi_{\eta_r}(0, 0) = 0$,*
3) *if $\bar{f}^c$ is transverse to the v.f. of the 4D chained system, then $f = \phi_{\eta_r}^{-1}(\bar{f}^c)$ is transversal to the v.f. $X_{1,\eta_r}$ and $X_2$.*

The third property implies that the matrix $H(\alpha, t)$ of relation (55) is invertible for any $(\alpha, t)$. The proof of this lemma is in [29]. It relies on the possibility of transforming, via a change of state and control variables, the kinematic equations of a car-like vehicle into a 4D chained system.

From the above lemma the design of a function $f$ such that the matrix $\bar{C}(\alpha, t)$ defined by (53) is invertible reduces essentially to the design of a TF for the 4D chained system. For instance, one can take the function $\bar{f}^c(\alpha) = f^c(\alpha_r)^{-1} f^c(\alpha)$ with $f^c$ given by (30) –the product here involved is the group operation associated with the 4D chained system. Moreover, choosing $\alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})'$ and $\varepsilon_{i1}$ $(i = 3, 4)$ as specified in Lemma 2 allows for the asymptotic stabilization of feasible reference trajectories. In this respect, Properties 1-2 in Lemma 3 are important because they ensure that $f = \phi_{\eta_r}^{-1}(\bar{f}_c)$ vanishes when $\bar{f}_c$ vanishes. With these choices for $f^c$ and

$\alpha_r$ one obtains:

$$\bar{f}^c(\alpha)=\begin{pmatrix}\varepsilon_{31}(s\alpha_3+1)+\varepsilon_{41}(s\alpha_4+1)\\ \varepsilon_{32}c\alpha_3\\ \frac{\varepsilon_{31}\varepsilon_{32}}{4}s2\alpha_3-\varepsilon_{42}c\alpha_4\\ \frac{\varepsilon_{31}^2\varepsilon_{32}}{6}(s\alpha_3)^2c\alpha_3-\frac{\varepsilon_{41}\varepsilon_{42}}{4}s2\alpha_4-\varepsilon_{31}\varepsilon_{42}s\alpha_3c\alpha_4\end{pmatrix}$$

Recall that the parameters $\varepsilon_{i,j}$ $(i,j=3,4)$ should also satisfy the inequalities (32). The corresponding function $f$ to be used in the control expression is thus

$$f(\alpha,t)=\phi^{-1}_{\eta_r(t)}(\bar{f}_c(\alpha)) \tag{58}$$

with $\phi^{-1}_{\eta_r}$, the inverse of $\phi_{\eta_r}$, given by

$$\phi^{-1}_{\eta_r}(\bar{x})=\begin{pmatrix}\bar{x}_1\\ \bar{x}_4+\eta_r\bar{x}_1^2/2\\ \arctan(\bar{x}_3+\eta_r\bar{x}_1)\\ \dfrac{\bar{x}_2+\eta_r}{\left(\sqrt{1+(\bar{x}_3+\eta_r\bar{x}_1)^2}\right)^3}-\eta_r\end{pmatrix} \tag{59}$$

From there the calculation of $A_\alpha$ and $A_t$ in (53) and (54) can be performed by using the relations

$$\begin{aligned}A_\alpha(\alpha,t)&=X(f_g(\alpha,t))^{-1}\frac{\partial f_g}{\partial\alpha}(\alpha,t)\\ &=X(f_g(\alpha,t))^{-1}\frac{\partial}{\partial\bar{x}}\phi^{-1}_{\eta_r(t)}(\bar{f}_c(\alpha))_{1,2,3}\frac{\partial\bar{f}^c}{\partial\alpha}(\alpha)\end{aligned}$$

with

$$\begin{aligned}\tfrac{\partial}{\partial\bar{x}}\phi^{-1}_{\eta_r}(\bar{x})_{1,2,3}&=\begin{pmatrix}1&0&0&0\\ \eta_r\bar{x}_1&0&0&1\\ \eta_r/d(\bar{x},\eta_r)&0&1/d(\bar{x},\eta_r)&0\end{pmatrix}\\ d(\bar{x},\eta_r)&=1+(\bar{x}_3+\eta_r\bar{x}_1)^2\end{aligned}$$

and

$$A_t(\alpha,t)=X(f(\alpha,t))^{-1}\frac{\partial}{\partial\eta_r}\phi^{-1}_{\eta_r(t)}(\bar{f}_c(\alpha))_{1,2,3}\ \dot{\eta}_r(t)$$

with

$$\frac{\partial}{\partial\eta_r}\phi^{-1}_{\eta_r}(\bar{x})_{1,2,3}=\left(0,\frac{\bar{x}_1^2}{2},\frac{\bar{x}_1}{d(\bar{x},\eta_r)}\right)'$$

### C. Determination of $\eta_r$

When addressing trajectory stabilization problems it is usually assumed that all reference trajectory components (the functions of time $g_r$ and $\eta_r$ in the present case) are specified. However, in the case of mobile robot applications it is often convenient to only specify the reference pose $g_r$ which corresponds to the desired situation of the vehicle's main body. An issue then is the determination of $\eta_r$. For feasible trajectories, provided that $u_{r,1}=\dot{x}_r\cos\theta_r+\dot{y}_r\sin\theta_r$ is different from zero, one has $\eta_r=\frac{\dot{\theta}_r}{u_{r,1}}$ (see Eq. (47)). This suggests, among other possibilities, the following choice

$$\eta_r=\frac{\dot{\theta}_r u_{r,1}}{u_{r,1}^2+\varepsilon} \tag{60}$$

with $\varepsilon$ a small positive number whose role is to ensure that *i)* $\eta_r$ is always well defined, in particular when the longitudinal velocity $u_{r,1}$ vanishes or when the motion of $g_r$ is not feasible for a car, and *ii)* $\eta_r$ is close to the ideal desired value $\frac{\dot{\theta}_r}{u_{r,1}}$ when the reference trajectory is feasible and $u_{r,1}\neq 0$.

### D. Control

To calculate the control (54) there remains to determine an auxiliary control vector $\bar{v}(z_g)$ which asymptotically stabilizes $z_g=e$ for the control system $\dot{z}_g=X(z_g)\bar{v}$. A possible choice yielding exponential stabilization is

$$\bar{v}(z_g)=X(z_g)^{-1}Kz_g \tag{61}$$

with $K$ a Hurwitz-stable matrix. Another possibility, as in the unicycle case, arises from the concern of limiting the control energy during the transient phase when $z_g$ converges to $e$ and, at the same time, of limiting the number of car maneuvers during this phase. As in the unicycle case let us rewrite the error-system's equation (52) as $\dot{z}_g=H(z_g,\alpha,t)\bar{\bar{u}}$ with

$$H(z_g,\alpha,t)=X(z_g)\mathrm{Ad}^X(f_g(\alpha,t))\bar{C}(\alpha,t) \tag{62}$$

$$\bar{\bar{u}}=\bar{u}-\bar{C}(\alpha,t)^{-1}(A_t(\alpha,t)+\mathrm{Ad}^X(\tilde{g}^{-1})v_r) \tag{63}$$

The idea is again to determine $\bar{\bar{u}}$ which minimizes at every time-instant the quadratic cost $\bar{\bar{u}}'W_1\bar{\bar{u}}$ under the constraint $z_g'H\bar{\bar{u}}+z_g'W_2z_g=0$, with $W_1$ and $W_2$ denoting two s.p.d. matrices. The fact that $\bar{\bar{u}}_1=u_1$ when $v_r\equiv 0$ and $A_t\equiv 0$ suggests to choose $W_1$ diagonal with the first diagonal entry larger than the others. The solution to this simple problem, previously derived in the unicycle case, is given by the relation (45) with $z_g$ replacing $z$, i.e.

$$\bar{\bar{u}}=-\frac{z_g'W_2z_g}{z_g'HW_1^{-1}H'z_g}W_1^{-1}H'z_g \tag{64}$$

The control $\bar{u}=(u_1,\dot{\alpha}')'$ is then calculated by using (63).

### E. Simulation results

For this simulation the car is represented as a tricycle whose length (distance between front and rear wheels) and width (distance between the two rear wheels) are equal to 2 (meters). The same reference trajectory as for the unicycle simulations is used. Note that the phases when it is either persistent (*pe*) or not persistent (*npe*) are the same as in the unicycle case. The reason is that, whenever the trajectory is feasible (i.e. when $v_{r,3}=0$), $v_{r,2}$ is equal to zero only when $v_{r,1}$ is itself equal to zero. The feedback control (63,64) which includes a monitoring of the transient phase (before the convergence of $z$ to zero) is used. The parameters chosen for this control are $W_1=\mathrm{diag}\{1,0.01,0.01\}$, $W_2=\mathrm{diag}\{1,1,1\}$, $k_\eta=5$. The TF parameters are $|\varepsilon_{31}|=0.14$, $\varepsilon_{32}=1.8$, $|\varepsilon_{41}|=0.8$, $\varepsilon_{42}=0.64$.

Figure 6 shows the time-evolution of the four components of the modified tracking error $z$. One can observe that, besides the initial transient phase of convergence of $z$ to zero, this error is also different from zero during short time-intervals. This is due to discontinuities of the TF which result from discontinuities of the term $\eta_r(t)$ involved in the TF calculation, themselves induced by discontinuities of the reference velocity $\dot{x}_r(t)$. Figures 7-10 attempt to visualize the vehicle's motion in the plane during the different phases of the reference trajectory.

The vehicle's real-time motion and the control performance are better visualized by downloading and viewing the corresponding video file *car.avi* contained in a compressed material file of 3.4MB in size available at *http://ieeexplore.ieee.org*.

## CONCLUSION

The stabilization of trajectories for nonholonomic systems has been addressed by using the framework of systems on Lie groups which is well adapted to the treatment of mechanical systems and their symmetries. In contrast with other methods dedicated to the stabilization of particular trajectories — fixed-points and persistent feasible trajectories—, the Transverse Function (TF) control approach here proposed aims in the first place at achieving the *practical* –by opposition to asymptotic– stabilization of reference trajectories regardless of their admissibility and other specific properties. Secondary objectives can then be considered. Asymptotic stabilization of persistent feasible trajectories is one of them, and an original contribution of the present study was to show that it can be achieved via a proper choice of the transverse function involved in the control law. Another one is the asymptotic stabilization of fixed-points. A preliminary study of this problem in [25] shows that solutions can again be obtained via the search for adequate *generalized* TFs. For instance, in the case of the 3D chained system, the TF considered in Section III-D can be used to this purpose provided that $\alpha_r$ is allowed to vary according to the simple law $\dot{\alpha}_r = k_\alpha(\alpha - \alpha_r)$, with $k_\alpha > 0$. The generalization of this result to higher dimensions is also addressed in [25], but it involves generalized TFs which are different from those considered in the present paper. Recall that when addressing these complementary issues it matters to keep in mind that the "perfect" controller capable of stabilizing any feasible reference trajectory asymptotically probably does not exist [20].

Possible extensions to the present study are numerous. One of them concerns experimental testing and validation. Whereas the TF control approach has already been experimented on a unicycle-type vehicle [1], [2], no experimentation on a car-like vehicle has been reported so far. Then, as mentioned above, the fine tuning of the properties of a TF controller much depends on the selected TF. The exploration of the possibilities offered via the choice of this function is still largely open. Concerning nonholonomic systems other than unicycles and cars the application and adaptation of the approach to systems like the rolling sphere [8], [31], [34], the general N-trailer [21], [38], and snake-like robots [13], [35] constitute, in our eyes, interesting and challenging research topics. In the case of the rolling sphere the solution proposed by the authors in [28] can probably be refined in order to improve the closed loop system's performance. The control of underactuated mechanical systems is also a domain for which encouraging initial results [22], [26] have been obtained and which calls for new developments.

## APPENDIX

### A. Recalls of differential relations on Lie groups

Let $g, h, \sigma$ denote elements of a Lie group $G$.

$$dL_{gh}(\tau) = dL_g(h\tau)dL_h(\tau) \tag{65}$$

$$dR_{gh}(\tau) = dR_h(\tau g)dR_g(\tau) \tag{66}$$

$$(dL_g(\tau))^{-1} = dL_{g^{-1}}(g\tau) \tag{67}$$

$$(dR_g(\tau))^{-1} = dR_{g^{-1}}(\tau g) \tag{68}$$

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h) \tag{69}$$

$$\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g^{-1}) \tag{70}$$

Relations (65) and (66) are obtained by application of the chain rule to the relations $L_{gh} = L_g \circ L_h$ and $R_{gh} = R_h \circ R_g$. Relations (67) and (68) are then deduced from (65) and (66) by setting $h = g^{-1}$ and using the fact that $L_e$ and $R_e$ are the identity operator on $G$. Relation (69) is deduced from the fact that, by (12) and the definition of $J_\sigma$,

$$\begin{aligned}\mathrm{Ad}(gh) &= dJ_{gh}(e) = d(J_g \circ J_h)(e) \\ &= dJ_g(e)dJ_h(e) = \mathrm{Ad}(g)\mathrm{Ad}(h)\end{aligned}$$

Relation (70) is deduced from (69) by setting $h = g^{-1}$ and using the fact that, by definition, $\mathrm{Ad}(e)$ is the identity operator.

Let $g_i$ $(i = 1, 2)$ denote two smooth curves on a Lie group $G$, and $v_i = (v_{i,1}, \ldots, v_{i,n})'$ denote the decomposition of $\dot{g}_i$ on a basis of the group's Lie algebra $\mathfrak{g}$, i.e.

$$\dot{g}_i = X(g_i)v_i := \sum_{k=1}^{n} X_k(g_i)v_{i,k}$$

with $X_1, \ldots, X_n$ a basis of left-invariant v.f. on $G$. Then,

$$\frac{d}{dt}(g_1^{-1}) = -dL_{g_1^{-1}}(e)dR_{g_1^{-1}}(g_1)\dot{g}_1 \tag{71}$$

$$= -dR_{g_1^{-1}}(e)dL_{g_1^{-1}}(g_1)\dot{g}_1 \tag{72}$$

$$= -dR_{g_1^{-1}}(e)X(e)v_1 \tag{73}$$

$$\frac{d}{dt}(g_1^{-1}g_2) = X(g_1^{-1}g_2)v_2 - dR_{g_1^{-1}g_2}(e)X(e)v_1 \tag{74}$$

$$= X(g_1^{-1}g_2)v_2 - dL_{g_1^{-1}g_2}(e)\mathrm{Ad}(g_2^{-1}g_1)X(e)v_1 \tag{75}$$

$$\frac{d}{dt}(g_1 g_2^{-1}) = dR_{g_2^{-1}}(g_1)dL_{g_1}(e)X(e)(v_1 - v_2) \tag{76}$$

$$\frac{d}{dt}(g_1 g_2^{-1}) = dL_{g_1 g_2^{-1}}(e)\mathrm{Ad}(g_2)X(e)(v_1 - v_2) \tag{77}$$

Relations (71) and (72) are obtained by differentiating the relation $g_1 g_1^{-1} = g_1^{-1}g_1 = e$ and using (67) and (68). Relation (73) is directly deduced from (72) and the fact that $\dot{g}_1 = X(g_1)v_1$, with $X_1, \ldots, X_n$ left-invariant. Relation (74) is then deduced from (73) and (66). Relation (75) is deduced from (74) and (12). Relation (76) is obtained by differentiating the equality $g_1 = (g_1 g_2^{-1})g_2$ and using (68). Finally, Relation (77) is deduced from (76), the fact that $\mathrm{Ad}(g_2) = \mathrm{Ad}(g_2 g_1^{-1})\mathrm{Ad}(g_1)$, (by (69)), and also the fact that

$$\begin{aligned}\mathrm{Ad}(g_1) &= dR_{g_1^{-1}}(g_1)dL_{g_1}(e) \\ &= dR_{g_2 g_1^{-1}}(g_1 g_2^{-1})dR_{g_2^{-1}}(g_1)dL_{g_1}(e)\end{aligned}$$

where the first equality comes from (12) and the second one from (66).

### B. Figures

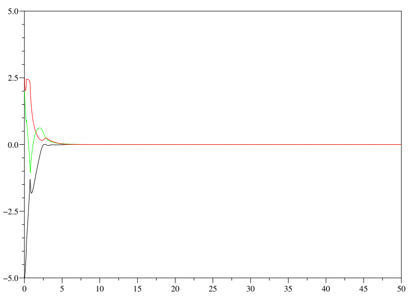


Fig. 1. Unicycle: $z_{1,2,3}$ vs. time

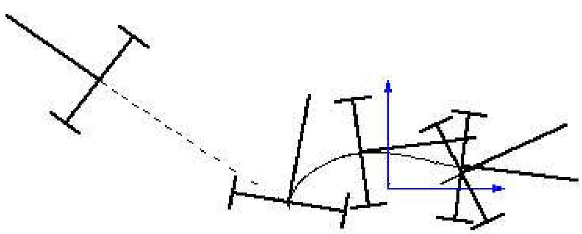

Fig. 2. Unicycle: Fixed reference $t \in [0s, 5s)$

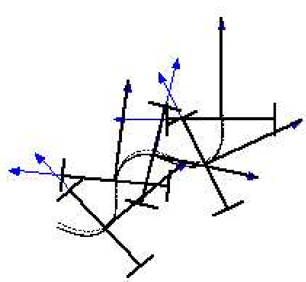

Fig. 3. Unicycle: Feasible trajectory with rapidly changing curvature $t \in [25s, 30s)$

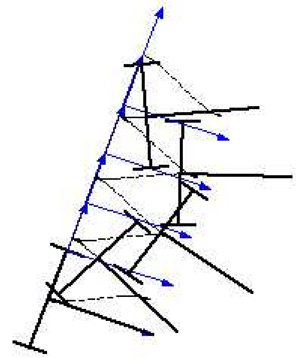

Fig. 4. Unicycle: Non-feasible lateral motion inducing maneuvers $t \in [30s, 35s)$

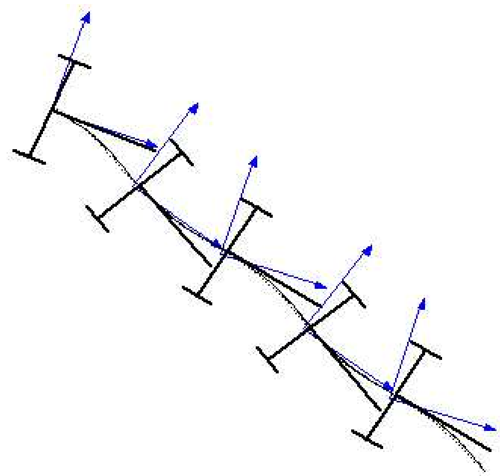

Fig. 5. Unicycle: Non-feasible motion not inducing maneuvers $t \in [40s, 45s)$

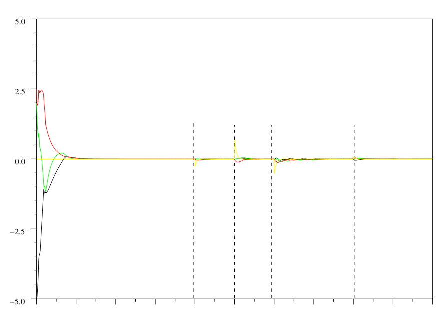


Fig. 6. Car: $z_{1,2,3,4}$ vs. time

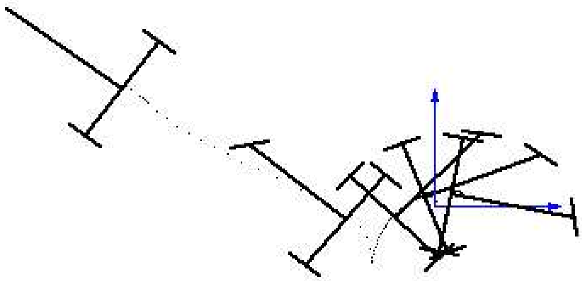

Fig. 7. Car: Fixed reference $t \in [0s, 5s)$

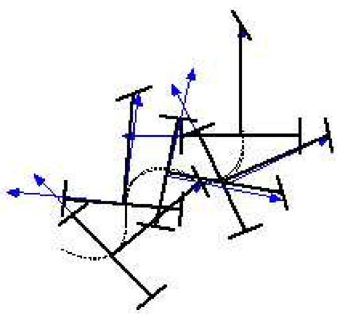

Fig. 8. Car: Feasible trajectory with rapidly changing curvature $t \in [25s, 30s)$

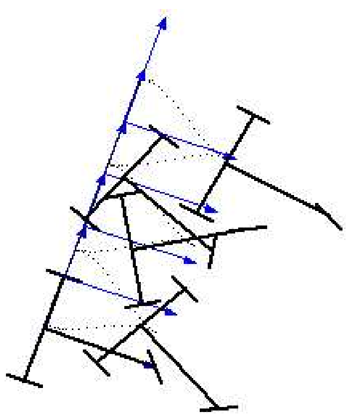

Fig. 9. Car: Non-feasible lateral motion inducing maneuvers $t \in [30s, 35s)$

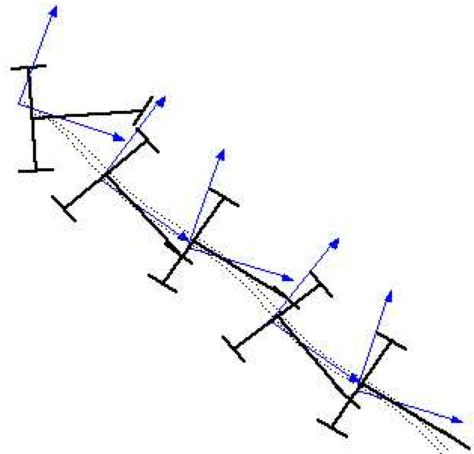

Fig. 10. Car: Non-feasible motion not inducing maneuvers $t \in [40s, 45s)$

## REFERENCES


[1] G. Artus, P. Morin, and C. Samson. Tracking of an omnidirectional target with a nonholonomic mobile robot. In *IEEE Conf. on Advanced Robotics (ICAR)*, pages 1468–1473, 2003.

[2] G. Artus, P. Morin, and C. Samson. Control of a maneuvering mobile robot by transverse functions. In *Symp. on Advances in Robot Kinematics (ARK)*, pages 459–468, 2004.

[3] A. Bicchi, A. Balluchi, D. Prattochizzo, and A. Gorelli. Introducing the "sphericle": an experimental testbed for research and teaching in nonholonomy. In *IEEE Conf. on Robotics and Automation*, pages 2620–2625, 1997.

[4] R.W. Brockett. Asymptotic stability and feedback stabilization. In R.W. Brockett, R.S. Millman, and H.J. Sussmann, editors, *Differential Geometric Control Theory*. Birkauser, 1983.

[5] R.W. Brockett. Robotic manipulators and the product of exponential formula. In P. A. Fuhrman, editor, *Proceedings of the mathematical theory of network and systems*, pages 120–129, 1984.

[6] C. Canudas de Wit, B. Siciliano, and G. Bastin, editors. *Theory of robot control*. Springer Verlag, 1996.

[7] W.L. Chow. Uber systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Ann.*, 117:98–105, 1939.

[8] H. Date, M. Sampei, M. Ishikawa, and M. Koga. Simultaneous control of position and orientation for ball-plate manipulation problem based on time-state control form. *IEEE Trans. on Robotics and Automation*, 20:465–479, 2004.

[9] W.E. Dixon, D.M. Dawson, E. Zergeroglu, and F. Zhang. Robust tracking and regulation control for mobile robots. *International Journal of Robust and Nonlinear Control*, 10:199–216, 2000.

[10] M. Fruchard, P. Morin, and C. Samson. A framework for the control of nonholonomic mobile manipulators. *The International Journal of Robotics Research*, 25(8):745–780, 2006.

[11] A. Halme, T. Schonberg, and Y. Wang. Motion control of a spherical mobile robot. In *IEEE Workshop on Advanced Motion Control*, pages 259–264, 1996.

[12] S. Hirose and A. Morishima. Design and control of a mobile robot with an articulated body. *International Journal of Robotics Research*, 9(2):99–114, 1990.

[13] M. Ishikawa, Y. Minami, and T. Sugie. Development and control experiment of the trident snake robot. In *IEEE Conf. on Decision and Control*, pages 6450–6455, 2006.

[14] O. Khatib and B. Siciliano. *Handbook of Robotics*. Springer, 2008.

[15] A. Koshiyama and K. Yamafuji. Design and control of an all-direction steering type mobile robot. *International Journal of Robotics Research*, 12(5):411–419, 1993.

[16] P.S. Krishnaprasad and D.P. Tsakiris. Oscillations, se(2)-snakes and motion control. In *IEEE Conf. on Decision and Control*, pages 13–15, 1995.

[17] N.E. Leonard and P.S. Krishnaprasad. Motion control of drift-free, left-invariant systems on Lie groups. *IEEE Trans. on Automatic Control*, 40:1539–1554, 1995.

[18] A.D. Lewis, J.P. Ostrowski, R.M. Murray, and J.W. Burdick. Nonholonomic mechanics and locomotion: the snakeboard example. In *IEEE Conf. on Robotics and Automation*, pages 2391–2397, 1994.

[19] W. Liu. An approximation algorithm for nonholonomic systems. *SIAM Journal on Control and Optimization*, 35:1328–1365, 1997.

[20] D.A. Lizárraga. Obstructions to the existence of universal stabilizers for smooth control systems. *Mathematics of Control, Signals, and Systems*, 16:255–277, 2004.

[21] D.A. Lizárraga, P. Morin, and C. Samson. Chained form approximation of a driftless system. Application to the exponential stabilization of the general N-trailer system. *International Journal of Control*, 74:1612–1629, 2001.

[22] D.A. Lizárraga and J.M. Sosa. Control of mechanical systems on Lie groups based on vertically transverse functions. *Mathematics of Control, Signals, and Systems*, 20(2):111–133, 2008.

[23] P. Morin and C. Samson. A characterization of the Lie algebra rank condition by transverse periodic functions. *SIAM Journal on Control and Optimization*, 40(4):1227–1249, 2001.

[24] P. Morin and C. Samson. Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Trans. on Automatic Control*, 48:1496–1508, 2003.

[25] P. Morin and C. Samson. Practical and asymptotic stabilization of chained systems by the transverse function control approach. *SIAM Journal on Control and Optimization*, 43(1):32–57, 2004.

[26] P. Morin and C. Samson. Control with transverse functions and a single generator of underactuated mechanical systems. In *IEEE Conf. on Decision and Control*, pages 6110–6115, 2006.

[27] P. Morin and C. Samson. *Handbook of Robotics*, chapter Motion control of wheeled mobile robots, pages 799–826. Springer, 2008.

[28] P. Morin and C. Samson. Stabilization of trajectories for systems on Lie groups. Application to the rolling sphere. In *IFAC World Congress*, pages 508–513, 2008.

[29] P. Morin and C. Samson. Trajectory tracking for nonholonomic systems. Theoretical background and applications. Research Report 6464, INRIA, 2008. available at https://hal.inria.fr/inria-00260694.

[30] P. Morin and C. Samson. Transverse function control of a class of non-invariant driftless systems. Application to vehicles with trailers. In *IEEE Conf. on Decision and Control*, pages 4312–4319, 2008.

[31] R. Mukherjee, M.A. Minor, and J.T. Pukrushpan. Motion planning for a spherical mobile robot: revisiting the classical ball-plate problem. *Journal of Dynamic Systems, Measurement, and Control*, 124:502–511, 2002.

[32] R.M. Murray, Z. Li, and S.S. Sastry. *A mathematical introduction to robotic manipulation*. CRC Press, 1994.

[33] H. Nijmeijer and A.J. Van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, 1991.

[34] G. Oriolo and M. Vendittelli. A framework for the stabilization of general nonholonomic systems with an application to the plate-ball mechanism. *IEEE Trans. on Robotics*, 21:162–175, 2005.

[35] J. Ostrowski and J. Burdick. The geometric mechanics of undulatory robotic locomotion. *International Journal of Robotics Research*, 17(7):683–701, 1998.

[36] O. J. Sørdalen. Conversion of the kinematics of a car with $n$ trailers into a chained form. In *IEEE Conf. on Robotics and Automation*, pages 382–387, 1993.

[37] H.J. Sussmann and W. Liu. Limits of highly oscillatory controls and approximation of general paths by admissible trajectories. In *IEEE Conf. on Decision and Control*, pages 437–442, 1991.

[38] M. Venditelli and G. Oriolo. Stabilization of the general two-trailer system. In *IEEE Conf. on Robotics and Automation*, pages 1817–1823, 2000.

[39] F.W. Warner. *Foundations of differential manifolds and Lie groups*. Springer Verlag, 1983.



**Pascal Morin** received the Maîtrise degree from Université Paris-Dauphine in 1990, and the Diplôme d'Ingénieur and Ph.D. degrees from Ecole des Mines de Paris in 1992 and 1996 respectively. He spent one year as a postdoctoral fellow in the Control and Dynamical Systems Department at the California Institute of Technology. Since 1997, he has been Chargé de Recherche at INRIA, France. His research interests include control of nonlinear systems and its applications to mechanical systems.



**Claude Samson** graduated from the Ecole Supérieure d'Electricité in 1977 and received his Docteur-Ingénieur and Docteur d'Etat degrees from the University of Rennes, in 1980 and 1983 respectively. After a postdoctoral stay at Stanford University, he joined INRIA in 1981 where he is presently Directeur de Recherche. His research interests are in Control Theory and its applications to Robotics. C. Samson is co-author, with M. Leborgne and B. Espiau, of the book *Robot Control. The Task-Function Approach* (Oxford University Press, 1991).