

A Control Approach for Thrust-Propelled Underactuated Vehicles and its application to VTOL drones

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Abstract—A control approach is proposed for a class of underactuated vehicles in order to stabilize reference trajectories either in thrust direction, velocity, or position. The basic modeling assumption is that the vehicle is propelled via a thrust force along a single body-fixed direction and that it has full torque actuation for attitude control (i.e. a typical actuation structure for aircrafts, Vertical Take-Off and Landing (VTOL) vehicles, submarines, etc.). Additional assumptions on the external forces applied to the vehicle are also introduced for the sake of control design and stability analyses. They are best satisfied for vehicles which are subjected to an external force field (e.g. gravity) and whose shape induces lift forces with limited amplitude, unlike airplanes but as in the case of many VTOL drones. The interactions of the vehicle with the surrounding fluid are often difficult to model precisely whereas they may significantly influence and perturb its motion. By using a standard Lyapunov-based approach, novel nonlinear feedback control laws are proposed to compensate for modeling errors and perform robustly against such perturbations. Simulation results illustrating these properties on a realistic model of a VTOL drone subjected to wind gusts are reported.

Index Terms—Thrust-propelled vehicle, underactuated system, nonlinear control, velocity stabilization, trajectory tracking, bounded nonlinear integrator, anti-windup.

I. INTRODUCTION

Airplanes, helicopters and other VTOL vehicles, blimps, rockets, hydroplanes, ships and submarines are generally *underactuated*. These vehicles are basically composed of a main body immersed in a fluid medium (air or water), and they are commonly controlled via i) a propulsive thrust force directed along a body-fixed privileged axis, and ii) a torque vector with one, two or three complementary independent components in charge of modifying the body's orientation. These vehicles are underactuated in the sense that, apart from the direction associated with the thrust force, the other possible direction(s) of displacement is (are) not directly actuated. Interestingly, the above-mentioned structural similitude has seldom been exploited to develop a general control framework for these vehicles. Various reasons can be proposed. For instance, there exist important differences between an airplane and a ship. The first vehicle evolves in air and 3D-space, whereas the other is (partly) immersed in water and essentially moves on a 2D-plane; the ambient fluid is not the same and it produces either

aerodynamic or hydrodynamic reaction forces with different properties and magnitude; gravity is not compensated by buoyancy in the case of an airplane, but lift-force effects are more systematic and preponderant; added-mass effects mostly concern ships, submarines, and blimps; etc.. Another probable reason is historical: aerospace and naval engineering communities involved in the control of these vehicles have not addressed common issues (the design of autopilots, for instance) in a coordinated manner, nor at the same time, nor with the same constraints (physical, economical, etc.), nor even with the same approaches. In this paper we consider the case of vehicles moving in 3D-space with four independent actuators (one force and three torques), knowing that six actuators would be necessary for full actuation. This is a common actuation structure for many underactuated vehicles evolving in 3D-space. Modeling assumptions concerning the environmental forces (drag, lift, gravity, etc) are introduced to simplify the control design and stability analyses. They basically infer i) that lift forces issued from the environment are limited in intensity and ii) the existence of an external force field. They are, for instance, satisfied by most wingless VTOL vehicles subjected to the action of gravity [9], [11], [19], [20]. Extension of the approach to vehicles whose motion relies on intense lift forces, like e.g. airplanes [1], remains to be worked out. As for the assumption about the existence of an external force field, it essentially serves to exclude the difficult case when the linearized system along the considered reference trajectory is not controllable. This case corresponds, e.g., to situations when buoyancy exactly annihilates the action of gravity, as this may happen in the 3D-case of submarines and blimps or in the 2D-case of a sea ship, and when the reference trajectory consists of a fixed “pose” (i.e. position and orientation). Specific (and still prospective) nonlinear techniques are then required to solve the stabilization problem (see e.g. [23]).

An important motivation of this work is related to robustness issues, which can be critical for the systems under consideration due to a combination of factors. First, the complexity of aero/hydro-dynamic effects impedes to obtain a precise dynamical model, valid in a large operating domain. Then, vehicles are often subjected to rapidly changing perturbations (wind gusts, sea currents, etc.) whose magnitude can be commensurable with the available actuation power. Finally, measurement/estimation errors of the vehicle's pose can be significant. Of course, these issues have long been investigated in the context of linear control (see e.g. [1], [8], [29], [34])

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and the references therein). More generally, all kinds of linear control methods (Nyquist-like techniques [7], [34], H_2 and H_∞ optimization [10], [21], LQR optimal control [18], [34], etc.) have been applied to these systems. They concern Single-Input-Single-Output (SISO) Control Augmented Systems (CAS) associated with the regulation of a single variable (pitch, yaw, roll angle, altitude, etc.) as well as Multi-Input-Multi-Output (MIMO) CAS. However, they only guarantee a limited domain of stability and are often based on restrictive assumptions (e.g. hovering regime without wind gusts for VTOL vehicles). For these reasons, nonlinear control design methods (feedback linearization, backstepping, sliding mode, etc.) have been increasingly investigated in recent studies [3], [12], [15], [20], [25], [28], [33]. Robustness issues are particularly important for light and/or small vehicles, like blimps or reduced-scale VTOL vehicles and AUVs (Autonomous Underwater Vehicles) because they are very sensitive to wind or current-induced perturbations. The last decades have witnessed an increasing interest in the construction and control of these vehicles. Let us mention the examples of the HoverEye [5], [6], [27], [28], X4-flyer [11], [35], iStar [19], or AVATAR [31] VTOL vehicles, the AURORA airship [2], [24], the ROGUE underwater glider [18], or the Minesniper AUV [30]. This interest is much related to the versatility of these systems for surveillance and inspection missions, and to the development of low-cost and low-weight embarked sensors (Inertial Measurement Unit, camera, etc.). However, nonlinear control studies focusing on robustness issues for these systems, like [2], [24], [28] for instance, are not numerous. The present paper is also a contribution to the design and analysis of robust nonlinear control laws.

In this paper, several control modes typically associated with different levels of motion autonomy are considered. Particular attention is paid to the three following problems: i) stabilization of (desired) reference thrust directions, ii) stabilization of reference linear velocities, and iii) stabilization of reference position-trajectories. The first and second problems relate typically to manual joystick-augmented-control situations, whereas the third one is associated with fully autonomous motion applications. At first glance, the proposed approach is reminiscent of methods described in [9], [28], [32] for the stabilization of hovering VTOL vehicles, based on the idea of: i) using the thrust force and the vehicle's orientation as control variables to stabilize the vehicle's velocity and/or position, and ii) applying a classical backstepping procedure or a high-gain controller to determine torque-inputs capable of stabilizing the desired orientation. Here, instead of the vehicle's orientation, we use its angular velocity as intermediary control input. This alleviates several difficulties associated with control inputs which belong to a compact manifold and enter the system's equation in a non-affine manner. It also allows to cast linear velocity and position control problems as natural extensions of the basic thrust direction control one. Another originality of this paper concerns unmodeled dynamics. It is well-known from Control Theory that integral correction constitutes an effective means to compensate for modeling, measurement, and/or estimation static errors (biases). However, it is also well-known that this type of correction may generate insta-

bility and windup problems. Many control design techniques addressing these problems have been proposed during the last decades, like linear Anti-Windup Bumpless Transfer (AWBT) schemes (see e.g. [17] and the references therein), or nonlinear nested saturations approaches (see e.g. [36]). AWBT control schemes can efficiently reduce windup effects, but their global stability is difficult to guarantee. Nested saturations approaches yield bounded correction terms, thus reducing effectively the risk of saturation of the actuators which could jeopardize the stability of the controlled system. However, this does not prevent the integral terms involved in the correction function to grow arbitrarily large, leading to slow desaturation and sluggish dynamics. The nonlinear integrator bounding technique proposed here deals with these problems and is another contribution of the paper. Finally, the way energy dissipation produced by motion reaction forces is exploited for the control design and the stability analyses constitutes to our knowledge a novel interpretation which justifies the use of simple models and supports observations made by other authors [30].

The paper is organized as follows. The notation and the dynamic modeling of the class of systems considered are recalled in Section II. Assumptions on the external forces applying to the system (gravity, aero/hydro-dynamic forces, etc.) are introduced and discussed in this section. By using a standard Lyapunov-based approach, initial control laws are derived in Section III under other assumptions which simplify and facilitate the exposition of the main lines of the approach. In Section IV, the controllers are modified in order to comply with more realistic assumptions. Simulation results for a particular model of a VTOL vehicle are described in Section V to illustrate the concepts. Finally, concluding remarks are given in Section VI.

II. NOTATION AND MODELING

A. Notation

In this paper, we focus on vehicles which can be modeled as rigid bodies immersed in a fluid. The following notation is used.

- G is the vehicle's center of mass, m its mass, and J its inertia matrix. Both \underline{m} and J are assumed to be constant
- $\mathcal{I}=\{O; \vec{i}_o, \vec{j}_o, \vec{k}_o\}$ is a fixed (inertial or Galilean) frame with respect to (w.r.t.) which the vehicle's absolute pose is measured. This frame is chosen as the NED frame (North-East-Down) with \vec{i}_o pointing to the North, \vec{j}_o pointing to the East, and \vec{k}_o pointing to the center of the earth. $\mathcal{B}=\{G; \vec{i}, \vec{j}, \vec{k}\}$ is a frame attached to the body. The vector \vec{k} is parallel to the thrust force axis. This leaves two possible and opposite directions for this vector. The direction here chosen (\vec{k} pointing downward nominally) is consistent with the convention used for VTOL vehicles.
- The vector of coordinates of G in the basis of the fixed frame \mathcal{I} is denoted as $x = (x_1, x_2, x_3)^T$ with the T-symbol used for the operation of transposition. Therefore, $\vec{OG} = x_1 \vec{i}_o + x_2 \vec{j}_o + x_3 \vec{k}_o$, a relation that we also write in a more concise way as $\vec{OG} = (\vec{i}_o, \vec{j}_o, \vec{k}_o)x$. The orientation of the body-fixed frame \mathcal{B} w.r.t. the inertial frame \mathcal{I} is represented by the rotation matrix R . The column

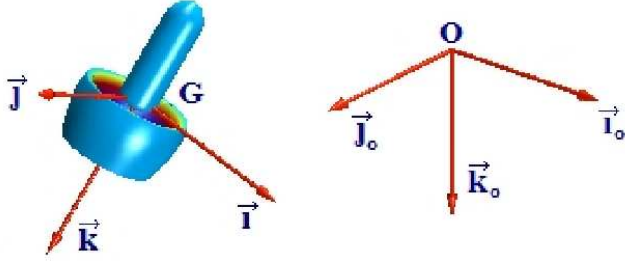


Fig. 1. Inertial and body-fixed frames.

vectors of R correspond to the vectors of coordinates of \vec{i} , \vec{j} , \vec{k} expressed in the basis of \mathcal{I} . The vector of coordinates associated with the linear velocity of G w.r.t. \mathcal{I} is denoted as $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)^T$ when expressed in the basis of \mathcal{I} , and as $v = (v_1, v_2, v_3)^T$ when expressed in the basis of \mathcal{B} , i.e. $\vec{v} = \frac{d}{dt} \vec{OG} = (\vec{i}_o, \vec{j}_o, \vec{k}_o) \dot{x} = (\vec{i}, \vec{j}, \vec{k}) v$. The angular velocity vector of the body-fixed frame \mathcal{B} relative to the fixed frame \mathcal{I} , expressed in \mathcal{B} , is denoted as $\omega = (\omega_1, \omega_2, \omega_3)^T$. The notation defined in the present item will also be used with the subscript “ r ” to denote reference trajectories, i.e. the trajectories to be stabilized. For example, \vec{v}_r denotes a reference velocity vector associated with the vehicle’s velocity vector \vec{v} . We assume that the reference trajectories are defined on $\mathbb{R}_+ = [0, +\infty)$.

- The ambient fluid velocity w.r.t. \mathcal{I} is denoted as $\vec{v}_f = (\vec{i}_o, \vec{j}_o, \vec{k}_o) \dot{x}_f = (\vec{i}, \vec{j}, \vec{k}) v_f$. The “apparent velocity” of the body \vec{v}_a is the difference between the velocity of G and the ambient fluid velocity, i.e. $\vec{v}_a = \vec{v} - \vec{v}_f$. One has also $\vec{v}_a = (\vec{i}_o, \vec{j}_o, \vec{k}_o) \dot{x}_a = (\vec{i}, \vec{j}, \vec{k}) v_a$, with $\dot{x}_a = \dot{x} - \dot{x}_f$ and $v_a = v - v_f$.

- $\{e_1, e_2, e_3\}$ denotes the canonical basis of \mathbb{R}^3 . $\forall u \in \mathbb{R}^3$, $S(u)$ denotes the skew-symmetric matrix associated with the cross product by u , i.e. $S(u)v = u \times v$, $\forall v \in \mathbb{R}^3$, with \times the cross product operation. The Euclidean norm in \mathbb{R}^n is denoted as $|\cdot|$, and the inner product as $\langle \cdot, \cdot \rangle$.

- A function $\eta : [t_o, +\infty) \rightarrow \mathbb{R}^p$ is u.b. (for *ultimately bounded*) by a constant c , if there exists a time t_1 such that $|\eta(t)| \leq c$, $\forall t \geq t_1$. An output $y = h(x, t) \in \mathbb{R}^p$ is u.u.b. (for *uniformly ultimately bounded*) by a constant c along the solutions to a differential equation $\dot{x} = f(x, t)$ if, $\forall (x_o, t_o)$, $y(\cdot) = h(x(\cdot, x_o, t_o), \cdot)$ is u.b. by c , where $x(\tau, x_o, t_o)$ denotes the solution at time τ with initial condition x_o at $t = t_o$.

B. System modeling

We consider mechanical systems with four control inputs: one force input T (also termed “thrust” input) along the body-fixed direction \vec{k} to create longitudinal motion, and three independent torque inputs to monitor the vehicle’s attitude. We assume that the thrust $\vec{T} = -T\vec{k}$ applies at a point that lies on, or close to, the axis $\{G; \vec{k}\}$, so that it does not create an important torque at G . The torque actuation is typically obtained via secondary propellers (X4-Flyer), rudders or flaps (HoverEye), control moment gyros (see [37]), etc. Complete torque actuation allows one to modify the vehicle’s attitude in order to direct the thrust at will. All external forces acting on the vehicle (gravity and buoyancy forces, added-mass forces,

and dissipative aerodynamic or hydrodynamic reaction forces) are summed up in a vector \vec{F}_e , so that the total resultant force applied to the vehicle is $\vec{F} = -T\vec{k} + \vec{F}_e$. In the absence of motion reaction forces exerted by the ambient fluid on the vehicle, only gravity, eventually counteracted by buoyancy forces of roughly constant magnitude, is present in \vec{F}_e . This force can then be modeled as a constant vector parallel to the $\{0; \vec{k}_o\}$ axis associated with the fixed frame \mathcal{I} . However, due to aerodynamic or hydrodynamic reaction forces, this vector generally depends on the apparent body velocity and acceleration (via added-mass effects), i.e. on $(\dot{x}_a, \ddot{x}_a, \omega, \dot{\omega})$ as well as on the vehicle’s attitude R . It may also depend on the vehicle’s position when the characteristics of the ambient fluid are not the same everywhere. For simplicity, this latter dependence will not be considered here. Moreover, whereas the dependence on accelerations is roughly linear, it is known that the intensities of motion reaction forces vary like the square of $|\dot{x}_a|$. Therefore, the intensity and direction of \vec{F}_e can vary considerably as soon as the vehicle’s velocity is modified significantly, or due to important modifications of the ambient environment (waves, wind, etc.). Modeling the various components of this function is, in general, time consuming and costly. This modeling effort is necessary for simulation purposes, and also for the optimization of the vehicles’ geometrical and mechanical characteristics. A model of \vec{F}_e can also be of use for control design purposes, but the knowledge of a precise and well-tuned model may not be as critically important as for simulation. Indeed, a well-designed feedback control is expected to grant robustness - in the sense of performance insensitivity- with respect to model inaccuracies. Furthermore, using on-line measurements or estimations of \vec{F}_e based on a crude model in the control can be preferable to using a sophisticated but nonetheless imperfect model of this force. In [14], an estimation of \vec{F}_e based on the measurement of the vehicle’s velocities and a high gain observer is proposed. Since there exists a variety of solutions to this problem, we henceforth assume that \vec{F}_e and its time-derivative are measured and/or estimated with “reasonably” good accuracy. Although the present paper focuses almost exclusively on control aspects we are aware that, for many applications, measurement/estimation issues are complementary and critically important.

Applying the fundamental theorem of Mechanics in the coordinates x yields

$$m\ddot{x} = -TRe_3 + F_e(\dot{x}, \ddot{x}, R, \omega, \dot{\omega}, t) \quad (1)$$

with $F_e \in \mathbb{R}^3$ the vector of coordinates of \vec{F}_e expressed in the inertial frame. By expressing the dynamics in the body-fixed frame \mathcal{B} , and by using Euler’s theorem of angular momentum, one obtains the following equations (see e.g. [8, Ch. 2],[34, Ch. 1])

$$\begin{aligned} \Sigma_1 : \begin{pmatrix} \dot{x} \\ m\dot{v} \\ \dot{R} \end{pmatrix} &= \begin{pmatrix} Rv \\ -mS(\omega)v - Te_3 + R^T F_e(\dot{x}, \ddot{x}, R, \omega, \dot{\omega}, t) \\ RS(\omega) \end{pmatrix} \\ \Sigma_2 : J\dot{\omega} &= -S(\omega)J\omega + \Gamma + \Gamma_e(\dot{x}, \ddot{x}, R, \omega, \dot{\omega}, t) \end{aligned} \quad (2)$$

with $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)^T$ the vector of torque inputs and Γ_e the external torque induced by the external forces. Equation (2) shows that the dynamical subsystem Σ_2 is fully-actuated. Exponential convergence of the angular velocity ω to any bounded desired reference value is then theoretically possible, especially when the external forces apply at points close to the center of mass G and $|\Gamma_e|$ is negligible. In this case, one may view ω as an intermediary control input. In practice, this corresponds to the classical decoupled control architecture between inner and outer loops. The inner control loop provides high gain stabilization of the vehicle's angular velocity based on direct measurement of the angular velocity from the Inertial Measurement Unit. The outer control loop uses pose measurement along with estimation or measurement of the linear velocity as sensor inputs, and the angular velocity set point and thrust intensity as control inputs. For many applications, the time-scale separation between the two loops is sufficient to ensure that the interaction terms can be ignored in the control design. Therefore, in the sequel we consider T and ω as the control inputs, and we focus on the control of the subsystem Σ_1 .

C. Assumptions

To simplify both the control design and the associated analyses we make some assumptions discussed hereafter.

Assumption 1 F_e depends only on the vehicle's linear velocity \dot{x} and the independent time variable t . Moreover, it is continuously differentiable with respect to these variables, and the functions $t \mapsto F_e(\dot{x}, t)$, $t \mapsto \frac{\partial F_e}{\partial \dot{x}}(\dot{x}, t)$, and $t \mapsto \frac{\partial F_e}{\partial t}(\dot{x}, t)$ are bounded on \mathbb{R}_+ , uniformly with respect to \dot{x} in compact sets.

The non-dependence of \vec{F}_e on the vehicle's attitude is physically justified when the aerodynamic (and/or hydrodynamic) forces do not depend on the vehicle's orientation, a property which depends essentially on the vehicle's shape. This assumption is clearly violated in the case of airplanes which are subjected to lift forces whose intensities are very sensitive to angles of attack, but it better holds in the case of VTOL vehicles, as exemplified by the "HoverEye" of Bertin Technologies Group. As for the non-dependence upon the angular velocity ω , Assumption 1 is better justified when i) the external forces apply at points close to the vehicle's center of mass, ii) motion reaction forces resulting from the vehicle's rotation can be neglected when compared to those produced by translational motion. Finally, the non-dependence on the acceleration variables \ddot{x} and $\dot{\omega}$ is justified when added-mass effects can be neglected. These effects can be ignored when the density of the body is much more important than that of the ambient fluid. The example of a dense spherical body whose center coincides with its center of mass can be used to concretize a physical situation for which Assumption 1 holds with a good approximation.

The following two complementary assumptions are much less restrictive than the previous one. However, they are very important for the control design and analyses presented in Section IV.

Assumption 2 There exist two real numbers $c_1 \geq 0, c_2 > 0$ such that

$$|F_e(\dot{x}, t)| \leq c_1 + c_2|\dot{x}|^2, \quad \forall(\dot{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+ \quad (3)$$

Assumption 3 There exist two real numbers $c_3 \geq 0, c_4 > 0$ such that

$$\dot{x}^T F_e(\dot{x}, t) \leq c_3|\dot{x}| - c_4|\dot{x}|^3, \quad \forall(\dot{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+ \quad (4)$$

Assumption 2 indicates that the intensity of \vec{F}_e cannot grow faster than the square of the intensity of the vehicle's velocity vector. This is consistent with common models of aerodynamic and hydrodynamic drag and lift forces (see e.g. [8, Ch. 2, 3], [34, Ch.2]). The constant c_1 allows to take into account the force of gravity, when it is active, and the action of perturbation forces produced by wind or sea-current. Assumption 3 follows from Assumption 2 and the "dissipativity", or "passivity", property of drag and lift forces. In particular, it indicates that for "large" velocities, the negative work of these forces increases like the cube of the body's apparent velocity, and thus becomes predominant when all other forces remain bounded.

Finally, the following assumption is made to avoid non-essential complications in the analyses.

Assumption 4 The reference velocity \vec{v}_r is bounded in norm on \mathbb{R}_+ by a constant \bar{v}_r , and its first and second order derivatives $\frac{d}{dt}\vec{v}_r$ and $\frac{d^2}{dt^2}\vec{v}_r$ are well-defined and bounded on this set.

III. BASICS OF THE CONTROL DESIGN

Using Assumption 1 the subsystem Σ_1 of System (2) can be rewritten as

$$\begin{aligned} \dot{x} &= Rv \\ \dot{v} &= -S(\omega)v - ue_3 + R^T\gamma_e(\dot{x}, t) \\ \dot{R} &= RS(\omega) \end{aligned} \quad (5)$$

with $\gamma_e(\dot{x}, t) := F_e(\dot{x}, t)/m$ called the "apparent acceleration", and $u := T/m$ and ω used hereafter as the control inputs.

This section is devoted to the stabilization of either the vehicle's thrust direction, or its linear velocity, or its position. The angular velocity ω_3 about the thrust axis is not involved in the realization of these control objectives, so that this degree of freedom can be used for complementary objectives and defined case-by-case, depending on the considered vehicle and application. *A priori*, there are infinitely many possibilities at this level, starting with the simplest choice $\omega_3 = 0$. In [26] (pages 105–108) this variable is determined in order to take advantage of lift forces associated with an asymmetric VTOL vehicle. In the sequel, to simplify the control design and the associated analyses we assume that ω_3 is well-defined and bounded on \mathbb{R}_+ .

A. Thrust direction control

Let $\gamma \in \mathbb{R}^3$ denote the normalized vector ($|\gamma| = 1$) of coordinates of the desired thrust direction expressed in the inertial frame \mathcal{I} . In practice this desired direction may be specified by a manual joystick. The objective is to stabilize the vehicle's thrust direction about the reference vector γ or, equivalently, to stabilize $R^T\gamma$ about e_3 . Define

$$\bar{\gamma} := R^T\gamma \quad (6)$$

and let $\tilde{\theta} \in (-\pi; \pi]$ denote the angle between the two unit vectors e_3 and $\bar{\gamma}$, so that $\cos \tilde{\theta} = \bar{\gamma}_3$, the third component of $\bar{\gamma}$. The control objective is also equivalent to the asymptotic stabilization of $\tilde{\theta} = 0$. The first control result of this paper is stated next.

Proposition 1 *Let k denote a strictly positive constant, and apply the control law*

$$\begin{aligned} \omega_1 &= -\frac{k\bar{\gamma}_2}{(1+\bar{\gamma}_3)^2} - \gamma^T S(Re_1)\dot{\gamma} \\ \omega_2 &= \frac{k\bar{\gamma}_1}{(1+\bar{\gamma}_3)^2} - \gamma^T S(Re_2)\dot{\gamma} \end{aligned} \quad (7)$$

to the system $\dot{R} = RS(\omega)$. Then the equilibrium point $\tilde{\theta} = 0$ of the controlled system is exponentially stable with domain of attraction equal to $(-\pi, \pi)$.

The proof of this proposition is given in Appendix B. In what follows we show how this controller can be extended to the linear velocity and position control problems.

Remark 1: The above control law (like those presented in the forthcoming sections) makes use of the feedforward term $\dot{\gamma}$, which is not always available in practice. Simulations with the VTOL model and numerical data of Section V have shown that good performance (in the sense of “small” ultimate tracking errors) is also obtained when $\dot{\gamma}$ in (7) is set equal to zero and its actual value is not too large. This can be justified rigorously using the Lyapunov function (33) of Appendix B.

Remark 2: To simplify the stability statement, the equilibrium set and domain of attraction have been expressed in term of the variable $\tilde{\theta}$. However, Proposition 1 can be stated without referring to $\tilde{\theta}$, by defining the equilibrium set as $\{R^* : R^*e_3 = \gamma\}$ and the associated domain of attraction as $\{R : \langle Re_3, \gamma(0) \rangle \neq -1\}$.

B. Velocity control

Let \dot{x}_r denote the reference velocity expressed in the inertial frame \mathcal{I} , \ddot{x}_r its time-derivative, and $\tilde{v} := R^T(\dot{x} - \dot{x}_r)$ the velocity error expressed in the body-fixed frame \mathcal{B} . Instead of defining γ as a reference unit vector as in the previous subsection, we now define

$$\gamma(\dot{x}, t) := \gamma_e(\dot{x}, t) - \ddot{x}_r(t) \quad (8)$$

One then obtains the following error model

$$\dot{\tilde{x}} = R\tilde{v} \quad (9a)$$

$$\dot{\tilde{v}} = -S(\omega)\tilde{v} - ue_3 + R^T\gamma(\dot{x}, t) \quad (9b)$$

$$\dot{R} = RS(\omega) \quad (9c)$$

with either $\tilde{x} := \int_0^t (\dot{x}(s) - \dot{x}_r(s)) ds$, the integral of the velocity error, or $\tilde{x} := x - x_r$, the position tracking error when a reference trajectory x_r is specified.

The problem of asymptotic stabilization of the linear velocity error $\dot{x} - \dot{x}_r$ to zero is clearly equivalent to the asymptotic stabilization of \tilde{v} to zero. Equation (9b) indicates that $\tilde{v} \equiv 0$ implies that

$$-ue_3 + R^T\gamma(\dot{x}, t) = 0 \quad (10)$$

As long as $\gamma(\dot{x}, t)$ is different from zero, one can define a locally unique thrust direction solution to the above equation. However, this solution cannot be prolonged by continuity at $\gamma = 0$. As a matter of fact, one can verify that this singularity corresponds to the case when the linearization of System (9b)–(9c) at any equilibrium point $(\tilde{v}, R) = (0, R^*)$ is not controllable. As explained in the introduction, this critical case is not the subject of the present paper. Beyond the technical difficulty associated with this case (which could be addressed in future studies), the main reason is that the vanishing of γ does not correspond to a generic situation for a large class of underactuated mechanical systems (those for which $\gamma_e(\dot{x}_r(t), t)$ is nominally different from zero). We thus essentially discard this issue here by assuming that

Assumption 5 *There exists a constant $\delta > 0$ such that $|\gamma(\dot{x}, t)| \geq \delta$ for all $(\dot{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+$.*

Although this assumption is restrictive, it simplifies the exposition of a basic and generic control design. In Section IV, however, we will weaken this assumption and propose an ad-hoc adaptation of the control design in order to ensure the well-posedness of the controller's expression and maintain a minimal control of the vehicle when $|\gamma|$ gets close to zero. When both Assumption 5 and relation (10) hold, using (6) one deduces that $\bar{\gamma} = \pm|\gamma|e_3$. Let $\tilde{\theta} \in (-\pi; \pi]$ denote the angle between the two unit vectors e_3 and $\frac{\bar{\gamma}}{|\bar{\gamma}|}$, so that $\cos \tilde{\theta} = \frac{\bar{\gamma}_3}{|\bar{\gamma}|}$. The control objective implies that either $\tilde{\theta} = 0$ (i.e. $\bar{\gamma} = |\gamma|e_3$) or $\tilde{\theta} = \pi$ (i.e. $\bar{\gamma} = -|\gamma|e_3$) must be asymptotically stabilized. The choice between these two equilibria is often made via simple physical considerations such as minimizing the energy consumption in relation to actuator's efficiency and vehicle's shape, or by taking into account the unilaterality of the thrust direction as in the case of most VTOL vehicles. Without loss of generality, we henceforth assume that the choice has been made to stabilize $\tilde{\theta} = 0$. Based on the above notation the second control result of this paper is stated in the next proposition.

Proposition 2 *Let k_1, k_2, k_3 denote strictly positive constants, and apply the control law*

$$\begin{aligned} u &= \bar{\gamma}_3 + |\gamma|k_1\tilde{v}_3 \\ \omega_1 &= -|\gamma|k_2\tilde{v}_2 - \frac{k_3|\gamma|\bar{\gamma}_2}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_1)\dot{\gamma} \\ \omega_2 &= |\gamma|k_2\tilde{v}_1 + \frac{k_3|\gamma|\bar{\gamma}_1}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_2)\dot{\gamma} \end{aligned} \quad (11)$$

to System (9). Suppose that Assumptions 1, 4, and 5 are satisfied. Then, for the subsystem (9b)–(9c), the equilibrium point $(\tilde{v}, \tilde{\theta}) = (0, 0)$ of the controlled system is asymptotically stable with domain of attraction equal to $\mathbb{R}^3 \times (-\pi, \pi)$.

The proof of this proposition is given in Appendix C. It is based on the use of the candidate Lyapunov function

$$V = \frac{1}{2}|\tilde{v}|^2 + \frac{1}{k_2} \left(1 - \frac{\tilde{\gamma}_3}{|\gamma|}\right) = \frac{1}{2}|\tilde{v}|^2 + \frac{1}{k_2}(1 - \cos \tilde{\theta}) \quad (12)$$

whose time-derivative is negative semi-definite along any solution of the controlled system.

C. Velocity control with integral term

For the stability and convergence analysis of the control (11), it is implicitly assumed that $\gamma(\dot{x}, t) = \gamma_e(\dot{x}, t) - \ddot{x}_r(t)$ is perfectly known. In practice however, due in particular to the difficulty of obtaining precise measures or estimates of F_e , the apparent acceleration γ_e is not known exactly, nor is γ therefore. It is well-known from the theory of linear control systems that integral correction terms can compensate for additive perturbations which, in the present case, may take the form of a constant bias in the measurement (or estimation) of γ_e . The objective of this subsection is to show that the control (11) can be modified in order to still ensure the convergence of $\dot{x} - \dot{x}_r$ to zero when such a bias is present. To this purpose let us introduce the following integral term

$$I_v(t) := \int_0^t (\dot{x}(s) - \dot{x}_r(s)) ds + I_0 \quad (13)$$

where I_0 is an arbitrary constant. Also, let h denote a smooth bounded strictly positive function defined on $[0, +\infty)$ such that, for some positive constants η, β ,

$$\forall s \in \mathbb{R}, \quad |h(s^2)s| < \eta \quad (14)$$

$$\forall s \in \mathbb{R}, \quad 0 < \frac{\partial}{\partial s}(h(s^2)s) < \beta \quad (15)$$

An example of such a function is $h : s \mapsto h(s) = \frac{\eta}{\sqrt{1+s}}$, with η a positive constant. Let $\hat{\gamma}_e$ denote the measure (or estimate) of γ_e and define now γ as follows (in replacement of relation (8))

$$\gamma := \hat{\gamma}_e - \ddot{x}_r + h(|I_v|^2)I_v \quad (16)$$

Proposition 3 *Apply the control law (11) to System (9) with γ defined by (16). Suppose that*

- i) *Assumptions 1, 4, and 5, with γ given by (16), are satisfied,*
- ii) *the measurement (or estimation) error $c := \gamma_e - \hat{\gamma}_e$ is constant,*
- iii) *$\lim_{s \rightarrow +\infty} h(s^2)s > |c|$.*

Then, for System (9b)–(9c) complemented with the equation $\dot{I}_v = R\tilde{v}$, there exists a constant vector $I_v^ \in \mathbb{R}^3$ such that the equilibrium point $(I_v, \tilde{v}, \tilde{\theta}) = (I_v^*, 0, 0)$ of the controlled system is asymptotically stable, with domain of attraction equal to $\mathbb{R}^3 \times \mathbb{R}^3 \times (-\pi, \pi)$.*

The proof is similar to the proof of Proposition 2. It is given in Appendix D.

Let us briefly comment on the role of the function h and its properties. The property (14) of h is introduced in order to limit, via (16), the influence of the integral I_v in the control action. However, Assumption iii) of Proposition 3 also points out that the upper-bound η associated with the choice of this

function should not be too small in order to compensate for a large estimation error c . On the other hand, in view of (16) a small value for η may reduce the risk of $|\gamma|$ evolving close to zero. This policy leads, for instance, to choose $\eta < g$ in the case when γ_e is essentially equal to the gravity acceleration, and the estimation error c and \ddot{x}_r are small compared to this acceleration. These considerations illustrate that a compromise has to be found, depending on the considered application.

D. Position control

The third control objective is the combined stabilization of the velocity error \tilde{v} (or $\dot{x} - \dot{x}_r$) and the position error $\tilde{x} = x - x_r$ to zero. A first solution to this problem is provided by the control proposed in the previous subsection since, by setting $I_0 = x(0) - x_r(0)$ in (13), one has $I_v = \tilde{x}$. Now, alike the velocity stabilization case, it can be useful (and even necessary) in practice to complement the control action with a position error integral correction term. A possibility consists in using a term proportional to the output z of a classical integrator of \tilde{x} (i.e. $\dot{z} = \tilde{x}$) in the control expression. However, this solution presents several drawbacks. For example, the integral correction term may grow very large and this may in turn cause large overshoots of the position tracking error. To avoid this problem, and also cope with actuator limitations, one must saturate the integral term. This can be done in many ways, some better than others. For instance, it is important to prevent the so-called desaturation (or windup) problem from occurring in order to not overly increase the system's time response. The solution proposed in this paper is based on a nonlinear dynamical extension yielding a type of bounded nonlinear integrator. More precisely, we denote z the solution to the following differential equation driven by \tilde{x}

$$\ddot{z} = -2k_z\dot{z} - k_z^2(z - \text{sat}_\Delta(z)) + k_z h_z(|\tilde{x}|^2)\tilde{x} \quad (17)$$

$(k_z > 0, z(0) = 0, \dot{z}(0) = 0)$

where h_z denotes a smooth bounded strictly positive function satisfying (14)–(15) for some positive constants η_z, β_z , and sat_Δ is a continuous ‘‘saturation function’’ characterized by the following properties, with Δ a positive number associated with this function,

- P1. sat_Δ is right-differentiable along any smooth curve and its derivative is bounded.
- P2. $\forall x \in \mathbb{R}^3$, if $|x| \leq \Delta$, $\text{sat}_\Delta(x) = x$.
- P3. $\exists \bar{\Delta} > 0$ such that $\forall x \in \mathbb{R}^3$, $|\text{sat}_\Delta(x)| \leq \bar{\Delta}$.
- P4. $\forall (c, x) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|c| < \Delta$,

$$|\text{sat}_\Delta(x + c) - c| \leq |x|.$$

A possible choice (for which $\Delta = \bar{\Delta}$) is the classical saturation function defined as

$$\text{sat}_\Delta(x) := x \min \left(1, \frac{\Delta}{|x|}\right) \quad (\Delta > 0) \quad (18)$$

One verifies from (17) that ultimate uniform upper-bounds of $|z|$, $|\dot{z}|$, and $|\ddot{z}|$ are $\bar{\Delta} + \eta_z/k_z$, $2(k_z\bar{\Delta} + \eta_z)$, and $6k_z(k_z\bar{\Delta} + \eta_z)$ respectively. Define

$$y := \tilde{x} + z \quad (19)$$

$$\bar{v} := \tilde{v} + R^T \dot{z} \quad (20)$$

$$\gamma := \hat{\gamma}_e - \ddot{x}_r + h(|y|^2)y + \ddot{z} \quad (21)$$

where $\hat{\gamma}_e$ denotes the measurement (or estimation) of γ_e , and h is a smooth bounded strictly positive function satisfying (14)–(15)¹ for some constants $\eta, \beta > 0$.

Proposition 4 *Let k_1, k_2, k_3 denote strictly positive constants. Apply the following control law*

$$\begin{aligned} u &= \bar{\gamma}_3 + |\gamma|k_1\bar{v}_3 \\ \omega_1 &= -|\gamma|k_2\bar{v}_2 - \frac{k_3|\gamma|\bar{\gamma}_2}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_1)\dot{\gamma} \\ \omega_2 &= |\gamma|k_2\bar{v}_1 + \frac{k_3|\gamma|\bar{\gamma}_1}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_2)\dot{\gamma} \end{aligned} \quad (22)$$

to System (9) with \bar{v}, γ , and y (which intervenes in the definition of γ) defined by (20), (21), and (19) respectively. Suppose that

- i) Assumptions 1, 4, and 5, with γ given by (21), are satisfied,
- ii) the measurement (or estimation) error $c := \gamma_e - \hat{\gamma}_e$ is constant,
- iii) $\lim_{s \rightarrow +\infty} h(s^2)s > |c|$,
- iv) $\Delta > |z^*|$, where z^* denote the unique solution to the equation $h(|z^*|^2)z^* = c$.

Then, for System (9) complemented with (17), the equilibrium point $(z, \dot{z}, \tilde{x}, \tilde{v}, \tilde{\theta}) = (z^*, 0, 0, 0, 0)$ of the controlled system is asymptotically stable, with domain of attraction equal to $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times (-\pi, \pi)$.

The proof of this proposition is given in Appendix E.

The role of the function h has been commented upon in the previous subsection. For position stabilization, we further remark that the property (14) of h bounds the contribution of the position error \tilde{x} in γ defined by (21), and thus also in the control inputs defined by (22). This limits the influence of large initial position errors on the control inputs intensity and reduces the risk of saturating the actuators. Note that the choice of h is still a matter of compromise. Let us comment on the role of the coefficient k_z . Equation (17) points out that k_z influences the rate of desaturation of z which can be observed, for instance, when $|z|$ is initially larger than Δ and $\tilde{x} = 0$. The larger k_z , the faster the desaturation and the smaller the influence of this integral action on the system's time response. On the other hand, since upper-bounds of $|\dot{z}|$ and $|\gamma|$ are proportional to k_z , a ‘‘small’’ value of k_z tends to limit the risk of saturating the actuators. A large value of k_z also increases the range interval of $|\gamma|$ and, subsequently, the risk of getting $|\gamma|$ close to zero (a value for which the control is no longer defined). The tuning of k_z is thus again a matter of compromise to be solved case-by-case depending on the considered application.

E. Control with unidirectional thrust

In many applications the thrust direction cannot be inverted. This means that only a positive (resp. negative) or null control u can be applied. For the control laws given in Propositions 2–4, this sign constraint is satisfied in the neighborhood of the

stabilized equilibrium point (since $u \approx |\gamma|$ and $|\gamma| > 0$ from Assumption 5). However, it is not satisfied in the entire domain of attraction of this equilibrium. The following proposition points out how the position control law (22) in Proposition 4 can be modified to comply with the constraint $u \geq 0$, without consequences on the stability issue.

Proposition 5 *Let k_1, k_2, k_3 denote strictly positive constants, and \bar{v} and γ as defined in Proposition 4. Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ denote a strictly increasing smooth function such that $\sigma(0) = 0$ and $\sigma(s) > -\frac{1}{k_1}, \forall s \in \mathbb{R}$. Apply the control law*

$$\begin{aligned} u &= |\gamma| + |\gamma|k_1\sigma(\bar{v}_3) \quad (\geq 0) \\ \omega_1 &= -|\gamma|k_2 \left(\bar{v}_2 - \frac{\bar{v}_3\bar{\gamma}_2}{|\gamma| + \bar{\gamma}_3} \right) - \frac{k_3|\gamma|\bar{\gamma}_2}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_1)\dot{\gamma} \\ \omega_2 &= |\gamma|k_2 \left(\bar{v}_1 - \frac{\bar{v}_3\bar{\gamma}_1}{|\gamma| + \bar{\gamma}_3} \right) + \frac{k_3|\gamma|\bar{\gamma}_1}{(|\gamma| + \bar{\gamma}_3)^2} - \frac{1}{|\gamma|^2}\gamma^T S(Re_2)\dot{\gamma} \end{aligned} \quad (23)$$

to System (9). Suppose that Assumptions i)–iv) of Proposition 4 are satisfied. Then the asymptotic stability result of Proposition 4 still holds.

The proof of this proposition is similar to the proof of Proposition 2. It is given in Appendix F. The proposed modification of the control law applies also to the control laws of Propositions 2 and 3 by simply replacing \bar{v} in the control expression (23) by \tilde{v} . A possible choice for the function σ is given, e.g., by

$$\sigma(s) = \frac{\alpha}{k_1} \tanh\left(\frac{k_1 s}{\alpha}\right), \text{ with } 0 < \alpha \leq 1$$

IV. CONTROL ROBUSTIFICATION

The results of the previous section rely upon the satisfaction of Assumption 5 which unconditionally guarantees the existence and local uniqueness of the desired thrust direction in the velocity and position control cases. For most underactuated vehicles, this assumption is too strong. Let us illustrate this on a simple example.

Example 1: (Spherical vehicle) Consider a spherical vehicle, with its center of mass coinciding with the sphere's center, submitted to the action of gravity, drag forces, and added-mass effects. The translational dynamics of the vehicle are given by (1) with $F_e(\dot{x}, \ddot{x}) = -c_a|\dot{x}|\dot{x} - m_a\ddot{x} + mge_3$, and c_a, m_a positive constant numbers associated with drag forces and added-mass effects respectively. This equation can be rewritten as (compare with (1)) $\bar{m}\ddot{x} = -TRe_3 + \bar{F}_e(\dot{x})$ with $\bar{m} = m + m_a$, and $\bar{F}_e(\dot{x}) = -c_a|\dot{x}|\dot{x} + mge_3$. The term $\gamma(\dot{x}, t)$ of equation (8) is thus given by $\gamma(\dot{x}, t) = \frac{\bar{F}_e(\dot{x})}{\bar{m}} - \ddot{x}_r = -\frac{c_a}{\bar{m}}|\dot{x}|\dot{x} + \frac{mg}{\bar{m}}e_3 - \ddot{x}_r$. When drag effects can be neglected (i.e. $\frac{c_a}{\bar{m}}|\dot{x}|\dot{x} \approx 0$), Assumption 5 is not satisfied when the reference acceleration vector \ddot{x}_r is equal to $\frac{mg}{\bar{m}}e_3$. As a matter of fact, the above relation points out that there always exists a velocity \dot{x} such that $\gamma(\dot{x}, t) = 0$.

Another example is provided by a ship drifting with the sea current at zero relative velocity. Indeed, the sum of external forces applied to the ship is equal to zero in this case. Even though one may hope that the set of ‘‘bad’’ velocities does not belong to the nominal operational domain of the vehicle, these examples indicate that, in some cases, Assumption 5 does not hold. Moreover, when $|\gamma| = 0$ the control is no longer defined.

¹Note that h can be different from h_z .

Now, to ensure *local* stability it is sufficient that γ does not vanish near the considered reference velocity \dot{x}_r . This suggests to replace Assumption 5 by the following weaker assumption.

Assumption 6 *There exists a constant $\delta > 0$ such that $|\gamma_e(\dot{x}_r(t), t) - \ddot{x}_r(t)| \geq \delta$ for all $t \in \mathbb{R}_+$.*

Under this assumption, the control laws proposed in Section III are locally well-defined and ensure *local* asymptotic stability of the desired reference velocity/position trajectory. This may be sufficient for many applications. However, for practical purposes one would like to ensure that the control calculation is always well-posed and that the tracking errors can never diverge explosively, whatever the adverse environmental conditions or poorly chosen reference trajectories for which γ approaches the null vector at some time-instant. Accordingly, the objective of this section is to modify the controllers of Section III in order to have the three following properties satisfied simultaneously:

- P1) local asymptotic stability when Assumption 6 is satisfied,
- P2) well-posedness of the expression of the control even when Assumption 6 is not satisfied,
- P3) global uniform ultimate boundedness of the system's velocities \dot{x} and ω even when Assumption 6 is violated.

The modifications are carried out for the velocity control objective of Proposition 2, but they are also valid for the other control laws proposed in Section III modulo straightforward transpositions which are shortly commented upon at the end of this section.

Property P2) is simply obtained by multiplying the unbounded terms $1/|\gamma|$ and $1/(|\gamma| + \bar{\gamma}_3)$ in the control expression (11) by an adequate function taking the value one inside a neighborhood of the reference trajectory and zero at $\gamma = 0$. For instance, one can use the class \mathcal{C}^1 function $\mu_\tau : [0, +\infty) \rightarrow [0, 1]$ defined by

$$\mu_\tau(s) = \begin{cases} \sin(\frac{\pi s^2}{2\tau^2}), & \text{if } s \leq \tau \\ 1, & \text{otherwise} \end{cases} \quad (24)$$

for some constant $\tau > 0$. This yields the modified control expressions

$$\begin{aligned} u &= \bar{\gamma}_3 + |\gamma|k_1\tilde{v}_3 \\ \omega_1 &= -|\gamma|k_2\tilde{v}_2 - \mu_\tau(|\gamma| + \bar{\gamma}_3)\frac{k_3|\gamma|\tilde{\gamma}_2}{(|\gamma| + \bar{\gamma}_3)^2} \\ &\quad - \mu_\tau(|\gamma|)\frac{1}{|\gamma|^2}\gamma^T S(Re_1)\dot{\gamma} \\ \omega_2 &= |\gamma|k_2\tilde{v}_1 + \mu_\tau(|\gamma| + \bar{\gamma}_3)\frac{k_3|\gamma|\tilde{\gamma}_1}{(|\gamma| + \bar{\gamma}_3)^2} \\ &\quad - \mu_\tau(|\gamma|)\frac{1}{|\gamma|^2}\gamma^T S(Re_2)\dot{\gamma} \end{aligned} \quad (25)$$

This modification does not forbid the satisfaction of Property P1). The fulfilment of Property P3) is more involved. It relies in the first place on the following observation, which is a direct consequence of the dissipativity of drag forces (i.e. Assumption 3).

Proposition 6 *Suppose that Assumption 3 is satisfied and that u is calculated according to a feedback law such that, for some constants β_1, β_2 ,*

$$|u| \leq \beta_1 + \beta_2|\dot{x}| \quad (26)$$

Then, the linear velocity \dot{x} of the controlled vehicle is u.u.b.. Moreover, under Assumption 1, $\gamma_e, \dot{\gamma}_e$ and the linear acceleration \ddot{x} are also u.u.b..

The proof of this proposition consists in calculating the time-derivative of $V = \frac{1}{2}|\dot{x}|^2$ and showing that it is negative when $|\dot{x}|$ exceeds a certain threshold (see [13] for details).

The objective is now to modify the expression (8) of $\gamma(\dot{x}, t)$ so that u , as given by (25), can satisfy inequality (26) without destroying the property of local asymptotic stability. To this purpose let $\overline{\text{sat}}_\Delta$ denote a continuous ‘‘saturation function’’ satisfying Properties P1, P2, P3 of the function sat_Δ and also the following property P4: There exists a continuous function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\forall(\xi, \gamma) \in \mathbb{R}^3 \times \mathbb{R}^3, \phi(\gamma) \leq 1$ and $\xi^T \overline{\text{sat}}_\Delta(\gamma) = \phi(\gamma)\xi^T \gamma$. A possible choice is the saturation function defined by (18).

Let $\gamma_d : \mathbb{R} \rightarrow \mathbb{R}^3$ denote any bounded function of class \mathcal{C}^1 whose derivative is also bounded. The role and choice of this function will be commented upon further, along with some examples. Define now γ as follows

$$\gamma(\dot{x}, t) := \gamma_d(t) + \overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}, t)) - \ddot{x}_r(t) \quad (27)$$

with

$$\gamma_{e,d}(\dot{x}, t) := \gamma_e(\dot{x}, t) - \gamma_d(t) \quad (28)$$

and M a positive real number the choice of which is discussed further. From (27), Assumption 4, the boundedness of γ_d , and the boundedness of the function $\overline{\text{sat}}_M$, it follows that there exists a finite value $Q > 0$ such that, whatever (\dot{x}, t) ,

$$|\gamma(\dot{x}, t)| \leq Q \quad (29)$$

Therefore, in view of the expression of u in (25), inequality (26) is now satisfied. Prior to stating the main stabilization result of this section we need to introduce some extra notation. Since γ_d is bounded by assumption, it follows from Assumptions 2 and 3 (recall that $\gamma_e = F_e/m$) and (28) that there exist constant numbers $\bar{c}_1 \geq 0, \bar{c}_2 > 0, \bar{c}_3 \geq 0, \bar{c}_4 > 0$ such that, $\forall(\dot{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$\begin{cases} |\gamma_{e,d}(\dot{x}, t)| & \leq \bar{c}_1 + \bar{c}_2|\dot{x}|^2 \\ \dot{x}^T \gamma_{e,d}(\dot{x}, t) & \leq \bar{c}_3|\dot{x}| - \bar{c}_4|\dot{x}|^3 \end{cases} \quad (30)$$

Consider the following polynomial in s : $P(s) := \bar{c}_4 s^3 - \bar{c}_2 \bar{\mathbf{v}}_r s^2 - \bar{c}_3 s - \bar{c}_1 \bar{\mathbf{v}}_r$, with $\bar{\mathbf{v}}_r > 0$. Since $\bar{c}_4 > 0$, there exists a number $\kappa(\bar{c}_i, \bar{\mathbf{v}}_r) \geq \bar{\mathbf{v}}_r$ such that $s \geq \kappa(\bar{c}_i, \bar{\mathbf{v}}_r) \Rightarrow P(s) \geq 0$.

The following theorem, proved in Appendix G, is the main result of this section.

Theorem 1 *Let k_1, k_2, k_3 denote strictly positive constants. Apply the control law (25) to System (9), with μ_τ and γ given by (24) and (27) respectively. Suppose that $0 < \tau < \delta$, where τ and δ are the constants involved in relation (24) and Assumption 6 respectively. Suppose that Assumptions 1, 2, 3, 4, and 6 are satisfied. Then,*

- 1) u and ω are well-defined and bounded along any solution of the controlled system,
- 2) \dot{x} is u.u.b. along any solution of the controlled system,
- 3) for the subsystem (9b)–(9c), the equilibrium point $(\tilde{v}, \tilde{\theta}) = (0, 0)$ of the controlled system is locally asymptotically stable if $M > \bar{c}_1 + \bar{c}_2 \bar{\mathbf{v}}_r$, with M the constant associated

with the function $\overline{\text{sat}}_M$ intervening in γ . Furthermore, if $M \geq \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$ and $|\gamma(\dot{x}, t)| \geq \tau \forall(\dot{x}, t)$, the attraction domain is equal to $\mathbb{R}^3 \times (-\pi, \pi)$.

By comparison with the control laws of Section III, the control (25) depends on extra design terms (τ , and the functions $\overline{\text{sat}}_M$ and γ_d) which can be tuned so as to maximize the domain of stability of the closed-loop system. Let us illustrate this tuning possibility in the case of the spherical vehicle already considered in Example 1.

Example 2: (Spherical vehicle, continued) To simplify we assume that the desired velocity \dot{x}_r is constant, i.e. $\ddot{x}_r = 0$. In this case, $\gamma_e(\dot{x}) = \gamma(\dot{x}) = \gamma_g + \gamma_{ae}(\dot{x})$, with $\gamma_g = (mg/\bar{m})e_3$ the gravity acceleration vector field, and $\gamma_{ae}(\dot{x}) = -(c_a/\bar{m})|\dot{x}|\dot{x}$ ($c_a > 0$) the acceleration vector associated with aerodynamic forces. Let us assume that the function $\overline{\text{sat}}_M$ is defined by (18), and consider two possible choices for γ_d (among others). First, let $\gamma_d = \gamma_g$. Then $\gamma = \gamma_g + \overline{\text{sat}}_M(\gamma_{e,d})$ and $\gamma_{e,d} = \gamma_{ae}$. Since the norm of γ_g is non-zero and constant, and $\overline{\text{sat}}_M$ is bounded by M , Assumption 6 is satisfied if $M < mg/\bar{m}$. In this case, $|\gamma(\dot{x}, t)| \geq mg/\bar{m} - M > 0$. Moreover, if $\tau < mg/\bar{m} - M$, the equilibrium $(\tilde{v}, \tilde{\theta}) = (0, 0)$ is “globally” asymptotically stable (i.e. the domain of attraction is $\mathbb{R}^3 \times (-\pi, \pi)$). However, imposing this inequality on M may not be compatible with the satisfaction of the condition $M \geq \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$ which guarantees the largest possible domain of attraction. Indeed, in this case one has $\bar{c}_1 = \bar{c}_3 = 0$, $\bar{c}_2 = \bar{c}_4 = c_a$, and $\kappa(\bar{c}_i, \bar{\mathbf{v}}_r) = \bar{\mathbf{v}}_r$, so that the condition $M \geq \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$ is now equivalent to $M \geq c_a \bar{\mathbf{v}}_r^2$. Therefore, the satisfaction of Assumption 6 and *global asymptotic stability* are guaranteed provided that $\sup|\dot{x}_r(t)| < \sqrt{mg/(\bar{m}c_a)}$. Now, to stabilize larger reference velocities which do not satisfy this inequality it is necessary to use values of M larger than mg/\bar{m} . However the positivity of $|\gamma(\dot{x}, t)|$ can no longer be guaranteed locally around any reference velocity. In this case, instead of choosing $\gamma_d = \gamma_g$, one might as well set $\gamma_d = 0$, so that $\gamma = \overline{\text{sat}}_M(\gamma_e) = \overline{\text{sat}}_M(\gamma_g + \gamma_{ae})$. With this choice the positivity of $|\gamma(\dot{x}, t)|$ is not unconditional, but it is satisfied in the neighborhood of any reference velocity such that $\dot{x}_r \neq \sqrt{mg/(\bar{m}c_a)}e_3$. From Theorem 1 *local asymptotic stability* is also obtained if $M \geq g + c_a \bar{\mathbf{v}}_r^2$.

Remark 3: The controllers of Propositions 3-5 can be modified in a similar way. Consider for example the position control law of Section III-D. One can define (compare with (21)) $\gamma := \gamma_d + \overline{\text{sat}}_M(\hat{\gamma}_{e,d}) - \ddot{x}_r + h(|y|^2)y + \ddot{z}$, with $\hat{\gamma}_{e,d} := \hat{\gamma}_e - \gamma_d$, and state stability results as in Theorem 1. The sole difference concerns the condition $M > \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$ of Theorem 1 which yields global asymptotic stability. It has to be replaced by the stronger condition $M > \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r + 2\eta_z))^2$.

V. SIMULATION RESULTS

This section illustrates the performance and robustness of the proposed controllers for a model of a VTOL vehicle similar to the “HoverEye” developed by Bertin Technologies Group (see [5], [6], [27], [28]). This vehicle, whose shape roughly corresponds to the one depicted on Figure 1, belongs to the class of “sit on tail” VTOL UAVs. It is symmetric

along a privileged axis taken as the axis $\{G; \vec{k}\}$. In the first approximation, its inertia matrix J is diagonal and $J \approx \text{diag}(J_1, J_1, J_2)$.

The system’s equations used in the simulations are given by (2) with (see [28] for details)

$$F_e = mge_3 + F_{ae} - \frac{1}{L}RS(e_3)\Gamma, \quad \Gamma_e = M_{ae} \quad (31)$$

where

- F_{ae} is the vector of coordinates of \vec{F}_{ae} expressed in the basis of the inertial frame \mathcal{I} , and \vec{F}_{ae} is the sum of all aerodynamic reaction forces (lift, drag, and momentum drag),
- M_{ae} is the torque induced by these forces,
- L is the distance between the plane of controlled fins and the vehicle’s center of mass.

Expressions of F_{ae} and M_{ae} are specified in [13]. Wind tunnel measurements and aerodynamic modeling for this class of VTOL vehicles have been reported in [27]. By setting

$$\gamma_e := F_e/m \quad (32)$$

the subsystem Σ_1 in (2) takes the form of System (5). However, Assumption 1 is violated because F_{ae} depends on the vehicle’s orientation and angular velocity, and also because Γ is related to the angular acceleration so that γ_e depends also on these variables. Discrepancies like this one between the ideal model used for the control design and the physical system represent an opportunity to test by simulation the robustness of the proposed controllers. The complete vehicle’s pose (i.e. x and R) is measured together with all velocity components (i.e. v and ω). The simulation results presented next have been obtained with the following estimated physical parameters of the vehicle: $m = 3 \text{ kg}$, $g = 9.8 \text{ ms}^{-2}$, $J_1 = 0.1 \text{ kgm}^2$, $J_2 = 0.03 \text{ kgm}^2$, $L = 0.2 \text{ m}$.

To test the robustness of the proposed controllers with respect to static modeling errors we assume that the real gravity acceleration is $g^* = 9.81 \text{ ms}^{-2}$, the real vehicle’s mass is $m^* = 3.2 \text{ kg}$, and the real vehicle’s inertia matrix is $J^* = \text{diag}(0.13, 0.13, 0.04)$.

Among the three control modes considered in the paper, position stabilization is the most advanced one and simulations are only presented for this mode. For the first subsystem Σ_1 the controller of Proposition 5, modified as proposed in Section IV, is applied. The desired yaw angular velocity is set to zero $\omega_{d,3} = 0$ and a high gain controller is applied to the second subsystem Σ_2 in order to stabilize the angular velocity at the desired value ω_d whose first two components are generated by the first controller. Note that the choice of a high gain controller is here justified by the fact that M_{ae} is neither measured nor estimated. The applied control torque is calculated according to $\Gamma = S(\omega)J\omega_d - JK_\omega(\omega - \omega_d)$, with K_ω a positive symmetric gain matrix here chosen diagonal.

The following gains and functions are used

- $k_1 = 0.24$, $k_2 = 0.08$, $k_3 = 12.8$, $K_\omega = \text{diag}(20; 20; 20)$,
- $h(s) = \frac{\beta}{\sqrt{1+\beta^2 s/\eta^2}}$ with $\beta = 1.28$ and $\eta = 12$,
- $\sigma(s) = \frac{\alpha}{k_1} \tanh\left(\frac{k_1 s}{\alpha}\right)$ with $\alpha = 0.9$,

- $k_z = 0.8$, $h_z(s) = \frac{\beta_z}{\sqrt{1+\beta_z^2 s/\eta^2}}$ with $\beta_z = 0.8$ and $\eta_z = 0.8$,
- sat_Δ and $\overline{\text{sat}}_M$ as given by (18) with $\Delta = 8$, $M = 50$.
- $\gamma_d = 0$, μ_τ as given by (24) with $\tau = 1$.

The gains k_1 , k_2 , k_3 , $h(0)$, k_z , $h_z(0)$ have been determined via a pole placement procedure performed on the linearized system of System (9)–System (17) at the equilibrium ($z = 0, \dot{z} = 0, \tilde{x} = 0, \tilde{v} = 0, R = I_3$) in the particular case of a reference trajectory consisting of a fixed point, with all external forces being neglected. Details can be found in [13]. Two simulation cases are reported.

Simulation 1: Stabilization at a stationary point.

The control objective is to stabilize the vehicle's center of mass G , initially resting at the position $x(0) = (8, 5, -8)^T$. The initial vehicle's attitude is given by $R(0) = I_3$. This corresponds to the equilibrium attitude associated with a fixed desired position in the absence of wind. The desired position is $x_r = (0, 0, 0)^T$. Initially there is no wind, but a horizontal wind step velocity $\hat{x}_f = (4, 0, 0)^T$ is introduced between the time-instants 30 s and 70 s , followed by a larger one ($\hat{x}_f = (8, 0, 0)^T$) thereafter. This simulation was devised to test the robustness of the proposed controller when neither measurement nor estimation of aerodynamic reaction forces is available. To this purpose we have used $\hat{\gamma}_e = g e_3$ in the control calculation, whereas the real value of γ_e is given by (31)–(32) with $g = g^*$ and $m = m^*$. It matters also to illustrate the role and importance of the integrator defined by (17). In this respect two control versions are used for comparison purposes. The first one does not incorporate a position integral action. This corresponds to setting the terms z , \dot{z} , and \ddot{z} equal to zero. The second one contains the integral action resulting from the calculation of z and its first and second order time-derivatives from (17). The evolution of the vehicle's position and attitude is shown on Figures 2 and 3. With both control versions, the position of the vehicle's center of mass G converges to a fixed position. However, in the no-integral action case (see Figure 2) the position error does not converge to zero due to estimation errors on the vehicle's physical parameters and poorly modeled aerodynamic reaction forces. Figure 3 shows that the incorporation of the proposed integral action makes this error converge to zero. Note that Assumption *iv*) of Proposition 5 ($\lim_{s \rightarrow +\infty} h(s^2)s > |c|$, with $c = \hat{\gamma}_e - \gamma_e$) must be satisfied to guarantee the stability of the controlled system and compensate for large wind-induced perturbations. When η , the upper-bound of $h(s^2)s$, is smaller than 10 and the wind velocity is "strong" ($\hat{x}_f = 8e_1$) –i.e. when modeling errors on external forces are very large– we have observed, in simulation, the divergence of the position error despite the integral action. This explains the use of a larger value of η (i.e. $\eta = 12$) in the reported simulations. Recall however that, as discussed in Subsection III-C, using a large value of η has the side drawback of increasing the risk of $|\gamma|$ evolving close to zero. It also contributes to increasing the magnitude of γ defined by (21), and thus also of the control inputs defined by (22). This in turn increases the risk of saturating the actuators, with known associated destabilizing effects. To comply with actuators power limitations a small value of η is preferable.

This in turn militates in favor of the on-line measurement or estimation of the apparent acceleration γ_e . In [14] a high gain observer of this force, based on the measurement of the vehicle's translational velocity \dot{x} and orientation R , and of the thrust intensity T , is proposed. Figure 4 shows simulation results of the controller with integral correction when using such an observer. Smaller values of η and Δ –the value associated with the function sat_Δ – (i.e. $\eta = 6$, $\Delta = 1$) are also applied. The improved tracking performance of this controller, which is also used in the next simulation case, shows the interest of complementing the integral correction action with the estimation of the apparent acceleration.

Simulation 2: Trajectory tracking with strong variable wind, large initial position error, and on-line estimation of aerodynamic forces.

The control objective is to track the following reference trajectory $x_r(t) = (10 \cos(\pi t/10), 10 \sin(\pi t/10), -t)^T$. The initial vehicle's position and attitude are given by $x(0) = (45, 50, -10)^T$ and $R(0) = I_3$ respectively. Integral correction in position is used. To test the robustness of the controller with respect to aerodynamic perturbations a "strong" variable wind is simulated with velocity intensity variations represented on Figure 6. The error of estimation of the apparent acceleration is also shown on this figure. Limitations of the actuators are also taken into account by saturating the applied thrust force and torque components according to the following inequality constraints $0 \leq T \leq 1.8m^*g^* = 56.5$, $|\Gamma_{i=1,2,3}| \leq 0.3TL$. The control results of Figure 5 illustrate the robustness of the controller with respect to strong and rapidly varying wind-induced perturbations and modeling errors. The tracking position errors decrease almost linearly from large initial values and remain small thereafter (see Figure 5.c). At the beginning of the simulation, due to the small value of η (i.e. $\eta = 6$) the thrust input remains unsaturated (see Figure 5.e) despite large initial position errors. The saturation occurring later on during short time intervals, as a consequence of strong wind-gusts, marginally affects the overall tracking performance.

VI. CONCLUSIONS

The present study attempts to set the foundations of a general approach to the control of a large family of thrust-propelled underactuated vehicles. Developing a control theory for vehicles seemingly as different as a VTOL, an underwater vehicle, or a space rocket may, at first glance, appears far-fetched and unrealistic. However, a closer look at the model equations of these systems brings evidence that the idea is technically relevant. The basic principle of the approach consists in monitoring the thrust direction in order to allow for the compensation of the resultant of external forces. Control laws conceived for incrementally complex objectives (ranging from joystick-augmented-control of the vehicle's attitude to autonomous trajectory tracking) have been derived, with the support of Lyapunov stability and convergence analyses. To cope with imprecise modeling and/or measurement of the forces acting on the vehicle, effective integral and anti-windup correction terms have been introduced, whereas this type of correction is often overlooked in nonlinear control studies.

On the other hand, the concern of genericity induced a certain number of simplifying assumptions. For instance, the existence of attitude control actuators enough powerful to overcome environmental perturbing torques was assumed, as well as the availability of accurate measurements/estimations of the vehicle's pose and velocity. Clearly the validity of these assumptions has to be assessed when considering an application on a physical system. For the genericity of the approach, it is important to extend this study in two directions. The first one concerns the assumption according to which environmental forces only depend on the vehicle's velocity (and the independent time-variable). This assumption needs to be relaxed because it is not realistic for a number of vehicles, like airplanes, for which drag and lift forces depend strongly on the angle of attack. The second direction concerns the assumption of non zero-crossing upon the so-called "apparent acceleration" –the resultant of external forces and desired accelerations. A route could consist in coupling the present approach with more involved, non-classical, control techniques aiming at the unconditional practical stability of the system [23]. Besides these conceptual developments, conducting experiments on physical systems is indispensable to consolidate the results of this study with respect to claims of robustness and performance in particular.

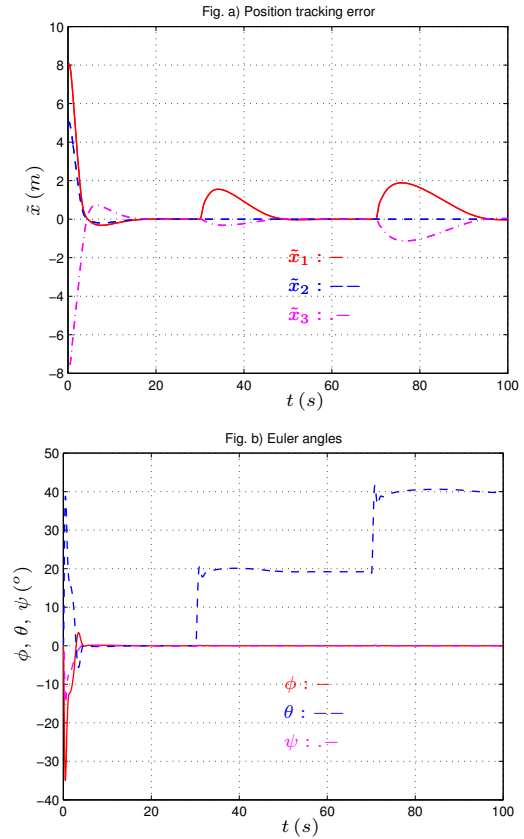


Fig. 3. Vehicle's pose vs. time, integral correction, $\eta = 12$, and $\Delta = 8$.

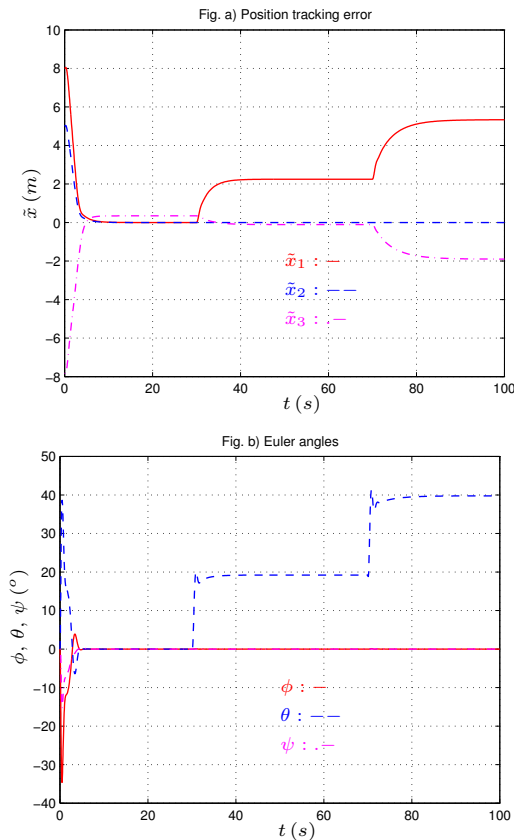


Fig. 2. Vehicle's pose vs. time, no integral correction, $\eta = 12$.

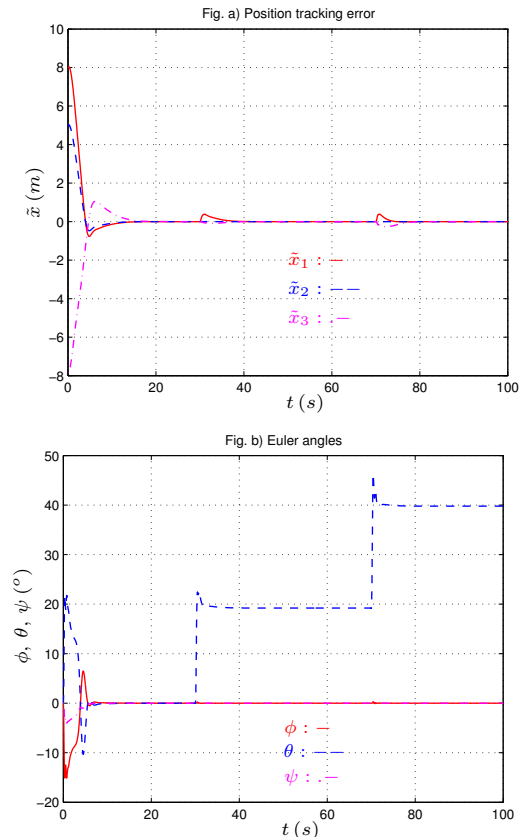


Fig. 4. Vehicle's pose vs. time, integral correction, $\eta = 6$, and $\Delta = 1$, on-line estimation of aerodynamic forces.

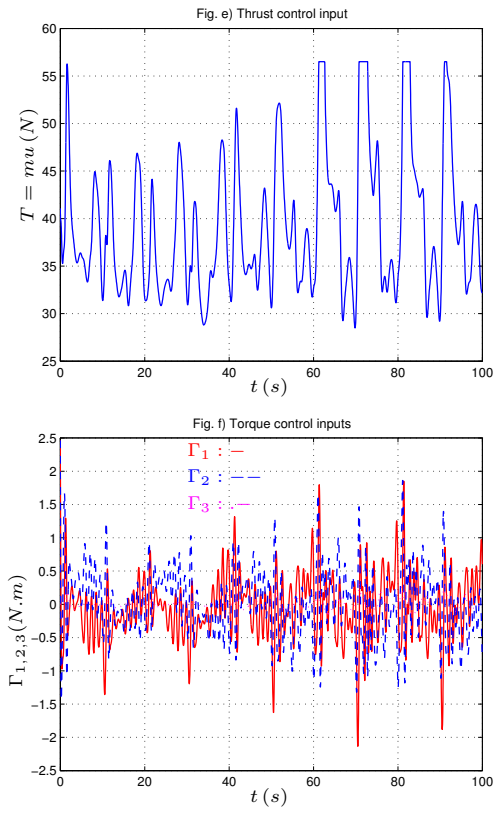
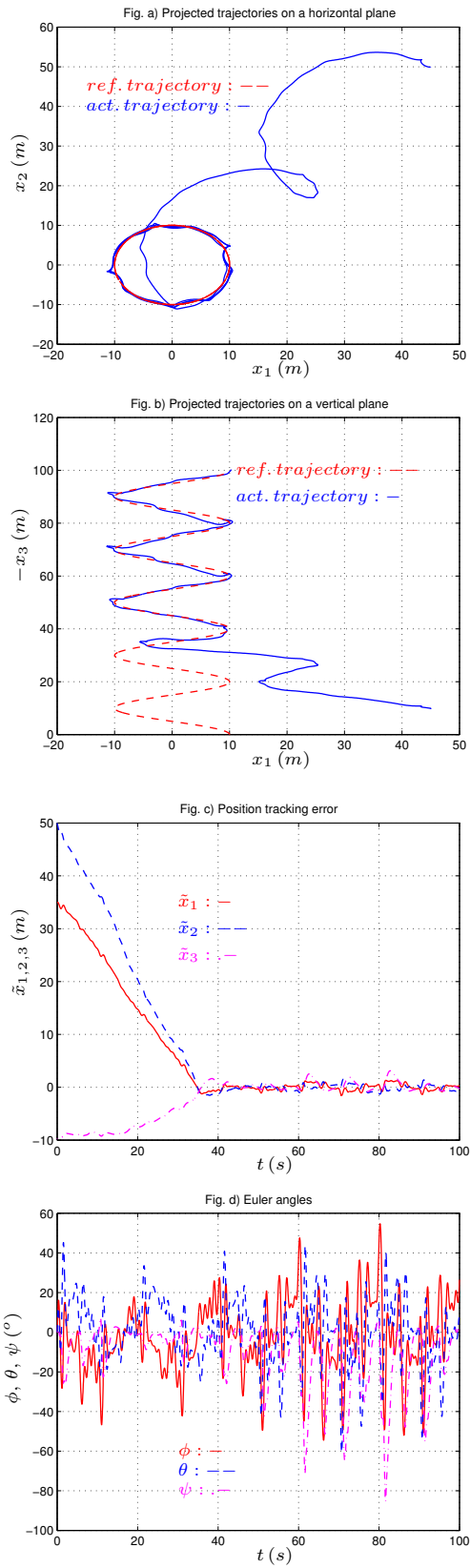


Fig. 5. Ascending reference spiral and vehicle's trajectory in the presence of strong variable wind and large initial position errors. Control with integral correction, $\eta = 6$, $\Delta = 1$, and on-line estimation of γ_e .

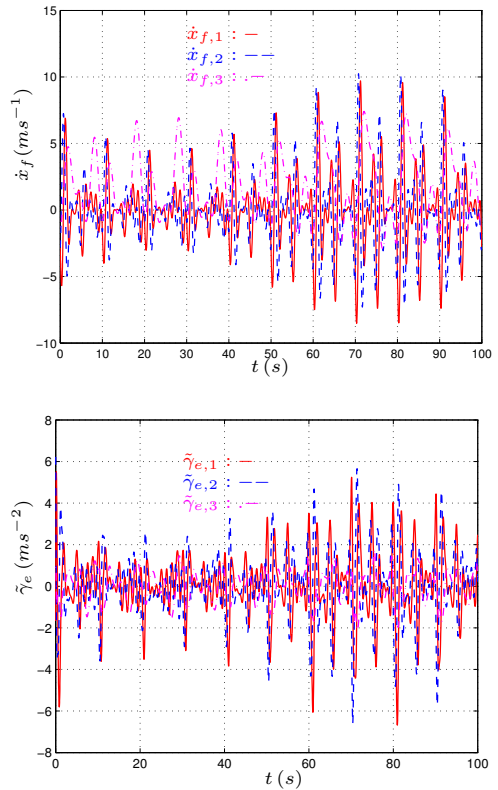


Fig. 6. Wind velocities vs. time. Error of estimation of the apparent acceleration ($\tilde{\gamma}_e = \gamma_e - \hat{\gamma}_e$).

APPENDIX

A. Technical lemma

Lemma 1 Let $\tilde{\gamma} = R^T \gamma$ with $\gamma \in \mathbb{R}^3$ a non-zero time-dependent vector and R the rotation matrix solution to (9c). Let $\tilde{\gamma}_{1,2}$ denote the vector $(\tilde{\gamma}_1, \tilde{\gamma}_2)^T$. Then,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\tilde{\gamma}}{|\tilde{\gamma}|} \right) &= \frac{1}{|\tilde{\gamma}|} S(\tilde{\gamma}) \left(\omega - \frac{1}{|\tilde{\gamma}|^2} R^T S(\gamma) \dot{\gamma} \right) \\ \frac{d}{dt} \left(1 - \frac{\tilde{\gamma}_3}{|\tilde{\gamma}|} \right) &= \frac{1}{|\tilde{\gamma}|} \tilde{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \frac{1}{|\tilde{\gamma}|^2} \begin{pmatrix} -\gamma^T S(Re_2) \dot{\gamma} \\ \gamma^T S(Re_1) \dot{\gamma} \end{pmatrix} \right] \end{aligned}$$

The proof of this technical lemma, based on elementary calculations, can be found in [13].

Lemma 2 (Barbalat, see e.g. [22]) Let $x(t)$ denote a solution to the differential equation $\dot{x} = a(t) + b(t)$ with $a(t)$ a uniformly continuous function. Assume that $\lim_{t \rightarrow +\infty} x(t) = c$ and $\lim_{t \rightarrow +\infty} b(t) = 0$, with c a constant value. Then, $\lim_{t \rightarrow +\infty} \dot{x}(t) = 0$.

The case $b = 0$ corresponds to the classical version of Barbalat's lemma (see e.g. [16]).

B. Proof of Proposition 1

Consider the following candidate Lyapunov function

$$V = 1 - \tilde{\gamma}_3 = 1 - \cos \tilde{\theta} \quad (33)$$

Differentiating V along the solutions of the system $\dot{R} = RS(\omega)$ and using Lemma 1 with $|\tilde{\gamma}| = 1$ one has

$$\dot{V} = \tilde{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \begin{pmatrix} -\gamma^T S(Re_2) \dot{\gamma} \\ \gamma^T S(Re_1) \dot{\gamma} \end{pmatrix} \right]$$

Using expressions (7) of ω_1, ω_2 one gets

$$\dot{V} = -k \frac{|\tilde{\gamma}_{1,2}|^2}{(1 + \tilde{\gamma}_3)^2} = -k \frac{1 - \tilde{\gamma}_3}{1 + \tilde{\gamma}_3} = -\frac{kV}{1 + \tilde{\gamma}_3} \leq -\frac{kV}{2} \leq 0 \quad (34)$$

This relation points out that V converges exponentially to zero. This in turn implies the exponential convergence of $\tilde{\theta}$ to zero. As for the stability of the equilibrium $\tilde{\theta} = 0$, it is a direct consequence of (33) and (34).

C. Proof of Proposition 2

It follows from the definition of $\tilde{\theta}$ that

$$\tan^2(\tilde{\theta}/2) = \frac{|\tilde{\gamma}_{1,2}|^2}{(|\tilde{\gamma}| + \tilde{\gamma}_3)^2} = \frac{|\gamma| - \tilde{\gamma}_3}{|\gamma| + \tilde{\gamma}_3} \quad (35)$$

Consider the candidate Lyapunov function V defined by (12). Differentiating V along the solutions of System (9b)-(9c) and using Lemma 1 one gets

$$\begin{aligned} \dot{V} &= \tilde{v}^T (-ue_3 + \tilde{\gamma}) \\ &+ \frac{1}{|\tilde{\gamma}| k_2} \tilde{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \frac{1}{|\tilde{\gamma}|^2} \begin{pmatrix} -\gamma^T S(Re_2) \dot{\gamma} \\ \gamma^T S(Re_1) \dot{\gamma} \end{pmatrix} \right] \\ &= \tilde{v}_3 (-u + \tilde{\gamma}_3) \\ &+ \frac{1}{|\tilde{\gamma}| k_2} \tilde{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \frac{1}{|\tilde{\gamma}|^2} \begin{pmatrix} -\gamma^T S(Re_2) \dot{\gamma} \\ \gamma^T S(Re_1) \dot{\gamma} \end{pmatrix} + |\tilde{\gamma}| k_2 \tilde{v}_{1,2} \right] \end{aligned}$$

with $\tilde{v}_{1,2} = (\tilde{v}_1, \tilde{v}_2)^T$. Substituting expressions (11) of u, ω_1, ω_2 and using (35) one obtains

$$\dot{V} = -|\tilde{\gamma}| k_1 \tilde{v}_3^2 - \frac{k_3}{k_2} \frac{|\tilde{\gamma}_{1,2}|^2}{(|\tilde{\gamma}| + \tilde{\gamma}_3)^2} = -|\tilde{\gamma}| k_1 \tilde{v}_3^2 - \frac{k_3}{k_2} \tan^2(\tilde{\theta}/2) \quad (36)$$

Since \dot{V} is negative semi-definite, the velocity error term \tilde{v} is bounded. The next step of the proof consists in showing that \dot{V} is uniformly continuous along every system's solution in order to deduce, by application of Barbalat's lemma (i.e. Lemma 2), the convergence of \tilde{v}_3 and $\tilde{\theta}$ to zero². To this purpose it suffices to show that \dot{V} is bounded. In view of (36), Assumption 5, and the boundedness of \tilde{v} , this condition is satisfied if $\gamma, \dot{\gamma}, \tilde{v}_3$, and $\frac{d}{dt} \tan^2(\tilde{\theta}/2)$ are bounded.

From Assumption 4, the boundedness of \tilde{v} , and the relation $\tilde{v} = R^T(\dot{x} - \dot{x}_r)$, it follows that \dot{x} is bounded. Therefore, using Assumptions 1 and 4 one deduces that γ_e, γ and $\tilde{\gamma}$ are also bounded, and that u (given by (11)) is also well-defined and bounded. This implies that \ddot{x} given by

$$\ddot{x} = -u Re_3 + \gamma_e(\dot{x}, t) \quad (37)$$

is bounded. Since $\dot{\gamma}_e(\dot{x}, t) = \frac{\partial \gamma_e}{\partial \dot{x}}(\dot{x}, t) \ddot{x} + \frac{\partial \gamma_e}{\partial t}(\dot{x}, t)$, it comes from Assumptions 1 and 4 and the fact that \dot{x} and \ddot{x} are bounded that $\dot{\gamma}_e$ and $\dot{\gamma}$ are also bounded. Let us now show that along each system's solution there exists $\varepsilon > 0$ such that

$$|\tilde{\theta}(t)| \leq \pi - \varepsilon, \quad \forall t \quad (38)$$

It follows from (11), (35), and Lemma 1 that

$$\begin{aligned} \frac{d}{dt} (1 - \cos \tilde{\theta}) &= -k_2 (\tilde{\gamma}_1 \tilde{v}_1 + \tilde{\gamma}_2 \tilde{v}_2) - k_3 \frac{|\gamma| - \tilde{\gamma}_3}{|\gamma| + \tilde{\gamma}_3} \\ &= -k_2 (\tilde{\gamma}_1 \tilde{v}_1 + \tilde{\gamma}_2 \tilde{v}_2) - k_3 \tan^2(\tilde{\theta}/2) \end{aligned}$$

Since $\tilde{\gamma}$ and \tilde{v} are bounded, there exists $\varepsilon_1 > 0$ such that $|\tilde{\theta}| > \pi - \varepsilon_1 \implies \frac{d}{dt} (1 - \cos \tilde{\theta}) < 0$. Equation (38) is thus satisfied with $\varepsilon = \min\{\varepsilon_1, \pi - |\tilde{\theta}(0)|\} (> 0)$. This implies the boundedness of $\tan(\tilde{\theta}/2)$ and also, from (35), of $1/(|\tilde{\gamma}| + \tilde{\gamma}_3)$. Along with Assumption 5 and the fact that $\tilde{v}, \gamma, \tilde{\gamma}, \dot{\gamma}$ are bounded, this ensures that the control inputs ω_1 and ω_2 , and thus ω , are well-defined and bounded. Since $\gamma, \tilde{v}, \omega, u$ are bounded, it follows from (9b) that $\dot{\tilde{v}}$ is also bounded. Finally, since both $\dot{\gamma}$ and ω are bounded, one deduces that $\dot{\tilde{\gamma}}$ is bounded. The boundedness of $\frac{d}{dt} \tan^2(\tilde{\theta}/2)$ then follows from (35) and from the boundedness of $1/(|\tilde{\gamma}| + \tilde{\gamma}_3)$. This concludes the proof of uniform continuity of \dot{V} and of the convergence of \tilde{v}_3 and $\tilde{\theta}$ to zero. Note from (35) that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ also converge to zero.

There remains to show that \tilde{v}_1 and \tilde{v}_2 converge to zero. By a direct calculation, one deduces from Lemma 1 and (11) that

$$\frac{d}{dt} \frac{\tilde{\gamma}_{1,2}}{|\tilde{\gamma}|} = a(t) + b(t) \quad (39)$$

with

$$\begin{aligned} a(t) &:= -k_2 \tilde{\gamma}_3 \tilde{v}_{1,2} - k_3 \tilde{\gamma}_3 \frac{\tilde{\gamma}_{1,2}}{(|\tilde{\gamma}| + \tilde{\gamma}_3)^2} \\ b(t) &:= \frac{1}{|\tilde{\gamma}|} \left(\omega_3 + \frac{1}{|\tilde{\gamma}|^2} \gamma^T S(Re_3) \dot{\gamma} \right) \begin{pmatrix} \tilde{\gamma}_2 \\ -\tilde{\gamma}_1 \end{pmatrix} \end{aligned}$$

²Note that LaSalle's theorem does not apply since the closed-loop dynamics is not autonomous.

It is straightforward to verify that $\dot{a}(t)$ is bounded (so that $a(t)$ is uniformly continuous) by using the boundedness of \dot{v} , $\bar{\gamma}$, $\dot{\gamma}$, and $1/(|\gamma| + \bar{\gamma}_3)$. Since $b(t)$ is not necessarily uniformly continuous (because of the terms ω_3 and $\dot{\gamma}$), the classical version of Barbalat's lemma does not apply. This explains the use of the slightly generalized version given in Lemma 2. Using the boundedness of ω_3 , Assumption 5, and the properties obtained previously (i.e. convergence of $\bar{\gamma}_{1,2}$ to zero and boundedness of $\dot{\gamma}$) one verifies that $b(t)$ converges to zero. Direct application of Lemma 2 to System (39) ensures the convergence of $\frac{d}{dt} \frac{\bar{\gamma}_{1,2}}{|\gamma|}$ to zero. Since $\bar{\gamma}_{1,2}$ converges to zero and $|\gamma| + \bar{\gamma}_3 > 0$, the convergence of \tilde{v}_1, \tilde{v}_2 to zero follows. As for the stability of the equilibrium $(\tilde{v}, \theta) = (0, 0)$, it is a direct consequence of relations (12) and (36).

D. Proof of Proposition 3

From the definition of γ , given by (16), equation (9b) can be rewritten as

$$\dot{\tilde{v}} = -S(\omega)\tilde{v} - ue_3 + R^T\gamma - R^T h(|I_v|^2)I_v + R^T c \quad (40)$$

Define the continuous function $f : s \mapsto h(s^2)s$. From (15), f is strictly increasing. Young's inequality (see [4]) then allows to establish the following relation

$$c^T I_v \leq |c| |I_v| \leq \int_0^{|I_v|} f(s) ds + \int_0^{|c|} f^{-1}(s) ds$$

with f^{-1} the inverse of f . This leads us to consider the following candidate Lyapunov function

$$V = \frac{1}{2} |\tilde{v}|^2 + \frac{1}{k_2} \left(1 - \frac{\bar{\gamma}_3}{|\gamma|}\right) + \int_0^{|I_v|} f(s) ds - c^T I_v + \int_0^{|c|} f^{-1}(s) ds \quad (41)$$

It is straightforward to verify that this function is positive and proper with respect to \tilde{v} . One verifies also that V is proper with respect to I_v by verifying that the Hessian matrix of V with respect to I_v is definite positive, i.e. $\frac{\partial^2 V}{\partial I_v^2} > 0$, using the condition (15) of the function h . Using (40), Lemma 1, and the relation $\dot{I}_v = R\tilde{v}$ one gets

$$\begin{aligned} \dot{V} &= \tilde{v}^T (-ue_3 + \bar{\gamma}) \\ &+ \frac{1}{|\gamma|k_2} \bar{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \frac{1}{|\gamma|^2} \begin{pmatrix} -\gamma^T S(Re_2)\dot{\gamma} \\ \gamma^T S(Re_1)\dot{\gamma} \end{pmatrix} \right] \end{aligned} \quad (42)$$

Substituting the control expression (11) into (42) one obtains that \dot{V} satisfies the equality (36). Then, the proof of convergence of (I_v, \tilde{v}, θ) to $(I_v^*, 0, 0)$ proceeds as for Proposition 2. Note in particular that the condition (15) of the function h is useful to ensure the boundedness of $\dot{\gamma}$, and that its combination with Assumption *iii*) of Proposition 3 implies the existence of a unique vector $I_v^* \in \mathbb{R}^3$ such that $h(|I_v^*|^2)I_v^* = c$. Furthermore, to prove that I_v converges to I_v^* one can apply Barbalat's lemma (i.e. Lemma 2) to (40) with $a(t) := -R^T h(|I_v|^2)I_v + R^T c$ and $b(t) := -S(\omega)\tilde{v} - ue_3 + R^T\gamma$. As for the stability of the equilibrium point $(I_v, \tilde{v}, \theta) = (I_v^*, 0, 0)$, it is a direct consequence of relation (41), the decrease of V along the system's solutions, and the fact that this point is the unique minimum of V .

E. Proof of Proposition 4

From (9b) and (20) it follows that the time-derivative of \bar{v} satisfies the equation $\dot{\bar{v}} = -S(\omega)\bar{v} - ue_3 + R^T\dot{z} + R^T(\gamma_e - \dot{x}_r)$ which can be rewritten as

$$\dot{\bar{v}} = -S(\omega)\bar{v} - ue_3 - R^T h(|y|^2)y + R^T\gamma + R^T c \quad (43)$$

with γ defined by (21). Consider the candidate Lyapunov function

$$V = \frac{1}{2} |\bar{v}|^2 + \frac{1}{k_2} \left(1 - \frac{\bar{\gamma}_3}{|\gamma|}\right) + \int_0^{|y|} f(s) ds - c^T y + \int_0^{|c|} f^{-1}(s) ds \quad (44)$$

where f is specified in the proof of Proposition 3 and $\bar{\gamma}$ is given by (6). As in the proof of Proposition 3, one verifies that V is positive and proper with respect to \bar{v} and y . Using (43), (22), and the relation $\dot{y} = R\bar{v}$ one gets

$$\dot{V} = -|\gamma|k_1 \bar{v}_3^2 - \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2} \quad (45)$$

From (17) one verifies that z , \dot{z} , and \ddot{z} are bounded. From (44) and (45) one deduces that y and \bar{v} are bounded. Since z and \dot{z} are bounded, the relations $y = \tilde{x} + z$ and $\bar{v} = \tilde{v} + R^T\dot{z}$ imply that \tilde{x} and \tilde{v} are also bounded. Then it follows from Property *P1* of the function sat_Δ , Property (15) of the function h_z , and System (17) that $z^{(3)}$ remains bounded. Denote z^* the unique solution to $h(|z^*|^2)z^* = c$. From there, with the same arguments as in the proof of Proposition 3, one deduces the convergence of (y, \bar{v}, θ) to $(z^*, 0, 0)$ and the stability of this equilibrium for the corresponding subsystem. Now, let $\bar{y} := y - z^*$, $\bar{z} := z - z^*$, $w := \dot{z}$, and $g_z(\bar{y}, \bar{z}) := h_z(|\bar{y} - \bar{z}|^2)(\bar{y} - \bar{z}) + h_z(|\bar{z}|^2)\bar{z}$. Note that $g_z(\bar{y}, \bar{z})$ is bounded and vanishes ultimately since \bar{y} converges to zero. Note also that $\tilde{x} = y - z = \bar{y} - \bar{z}$. Then, System (17) can be rewritten as

$$\begin{cases} \dot{\bar{z}} &= w \\ \dot{w} &= -2k_z w - k_z^2 \bar{z} + k_z^2 (\text{sat}_\Delta(\bar{z} + z^*) - z^*) \\ &\quad - k_z h_z(|\bar{z}|^2)\bar{z} + k_z g_z(\bar{y}, \bar{z}) \end{cases}$$

or in the more compact form

$$\dot{Z} = F(Z) + G(\bar{y}, Z) \quad (46)$$

with $Z := (\bar{z}, w)^T$, $F(Z) := (w, -2k_z w - k_z^2 \bar{z} + k_z^2 (\text{sat}_\Delta(\bar{z} + z^*) - z^*) - k_z h_z(|\bar{z}|^2)\bar{z})^T$, and $G(\bar{y}, Z) := (0, k_z g_z(\bar{y}, \bar{z}))^T$ (a "perturbation" which vanishes ultimately). One verifies that $Z = 0$ is the globally asymptotically stable point of the system $\dot{Z} = F(Z)$ by considering the candidate Lyapunov function

$$U = \frac{1}{2k_z} \int_0^{|\bar{z}|^2} h_z(s) ds + \frac{1}{2} |\bar{z}|^2 + \frac{1}{2} \left| \bar{z} + \frac{w}{k_z} \right|^2 \quad (47)$$

Indeed, differentiating U along the solution of the system $\dot{Z} = F(Z)$ and using Property *P4* of the function sat_Δ and Assumption *iv*) of Proposition 4 one obtains

$$\begin{aligned} \dot{U} &= -h_z(|\bar{z}|^2)|\bar{z}|^2 - k_z |\bar{z}|^2 \\ &\quad - k_z \left(\left| \bar{z} + \frac{w}{k_z} \right|^2 - (\bar{z} + (\text{sat}_\Delta(\bar{z} + z^*) - z^*))^T \left(\bar{z} + \frac{w}{k_z} \right) \right) \\ &\leq -h_z(|\bar{z}|^2)|\bar{z}|^2 - k_z |\bar{z}|^2 - k_z \left| \bar{z} + \frac{w}{k_z} \right|^2 + 2k_z |\bar{z}| \left| \bar{z} + \frac{w}{k_z} \right| \end{aligned} \quad (48)$$

Since z is bounded, \bar{z} is also bounded from its definition. As a consequence, there exists $\delta_z > 0$ such that $h_z(|\bar{z}|^2) > \delta_z$. This, along with relation (48), implies the existence of some positive constants $\alpha_z, \alpha_w, \alpha_u$ such that

$$\dot{U} \leq -\alpha_z |\bar{z}|^2 - \alpha_w \left| \bar{z} + \frac{w}{k_z} \right|^2 \leq -\alpha_u U \quad (49)$$

Therefore, $Z = 0$ is an exponentially stable equilibrium of the unperturbed system $\dot{Z} = F(Z)$. Since $G(\bar{y}, Z)$ converges to zero, this in turn implies that the solutions to System (46) converge to zero. The convergence of (\bar{z}, \dot{z}) to $(0, 0)$ follows. Then, the convergence of $\bar{y} = \bar{z} + \tilde{x}$ to zero (proved previously) implies that \tilde{x} converges to zero. Finally, since $\bar{v} = \tilde{v} + R^T \dot{z}$ and \dot{z} converge to zero, \tilde{v} also converges to zero.

Let us finally establish the stability of the equilibrium point $(z, \dot{z}, \tilde{x}, \tilde{v}, \theta) = (z^*, 0, 0, 0, 0)$. From (44) and (45) $(\bar{v}, \bar{y}, \theta) = (0, 0, 0)$ is a stable equilibrium of the controlled system, and from (47) and (49) $Z = 0$ is an asymptotically stable equilibrium point of the system $\dot{Z} = F(Z)$. Since the function G in (46) is continuous and identically equal to zero when $\bar{y} = 0$, one deduces that $(\bar{v}, \bar{y}, \theta, Z) = (0, 0, 0, 0)$ is also a stable equilibrium of the controlled system. The stability of the equilibrium $(z, \dot{z}, \tilde{x}, \tilde{v}, \theta) = (z^*, 0, 0, 0, 0)$ follows directly.

F. Proof of Proposition 5

The positivity of the thrust input u results from the definition of u given in (23) and the assumptions on σ in Proposition 5. Consider now the candidate Lyapunov function given by (44). Using (43), Lemma 1, and the control expression (23) one gets $\dot{V} = -|\gamma|k_1\sigma(\bar{v}_3)\bar{v}_3 - \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2}$. From this equality the proof proceeds like the proof of Proposition 4.

G. Proof of Theorem 1

The proof relies on the following technical lemma.

Lemma 3 *Let $\gamma_{e,d}$ as defined by (28). If $M \geq \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$, then $\forall(\dot{x}, t)$,*

$$\tilde{v}^T R^T (\gamma_{e,d}(\dot{x}, t) - \overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}, t))) \leq 0 \quad (50)$$

Proof: If $|\gamma_{e,d}(\dot{x}, t)| < M$, Property P2 of the saturation function $\overline{\text{sat}}_M$ implies that $\gamma_{e,d}(\dot{x}, t) = \overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}, t))$, and the result follows. If $|\gamma_{e,d}(\dot{x}, t)| \geq M$, it follows from (30) and the choice of M that $|\dot{x}| > \kappa(\bar{c}_i, \bar{\mathbf{v}}_r)$. Then, by using Property P4 of the saturation function $\overline{\text{sat}}_M$, relation (30), the relations $\phi(\gamma_{e,d}(\dot{x}, t)) \leq 1$ and $R\tilde{v} = \dot{x} - \dot{x}_r$ one gets

$$\begin{aligned} & \tilde{v}^T R^T (\gamma_{e,d}(\dot{x}, t) - \overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}, t))) \\ &= (1 - \phi(\gamma_{e,d}(\dot{x}, t))) (\dot{x} - \dot{x}_r)^T \gamma_{e,d}(\dot{x}, t) \\ &\leq (1 - \phi(\gamma_{e,d}(\dot{x}, t))) (\dot{x} \gamma_{e,d}(\dot{x}, t) + \bar{\mathbf{v}}_r |\gamma_{e,d}(\dot{x}, t)|) \\ &\leq -(1 - \phi(\gamma_{e,d}(\dot{x}, t))) (\bar{c}_4 |\dot{x}|^3 - \bar{c}_2 \bar{\mathbf{v}}_r |\dot{x}|^2 - \bar{c}_3 |\dot{x}| - \bar{c}_1 \bar{\mathbf{v}}_r) \end{aligned}$$

From the definition of the function κ , the inequality $|\dot{x}| > \kappa(\bar{c}_i, \bar{\mathbf{v}}_r)$ implies that $\bar{c}_4 |\dot{x}|^3 - \bar{c}_2 \bar{\mathbf{v}}_r |\dot{x}|^2 - \bar{c}_3 |\dot{x}| - \bar{c}_1 \bar{\mathbf{v}}_r \geq 0$. Inequality (50) follows. ■

From (25) and (29) relation (26) holds true for some positive constants β_1, β_2 . Property 2 of Theorem 1 (together with the completeness of the system's solutions) then directly

follows by application of Proposition 6. Since Assumption 1 holds, Proposition 6 implies also that $\gamma_e, \dot{\gamma}_e, \dot{x}$ are bounded. Since $\dot{\gamma}_d(t)$ is bounded, one deduces from (28) that $\dot{\gamma}_{e,d}(\dot{x}, t)$ remains bounded. Combined with Assumption 4, relation (27), and Property P1 of the function $\overline{\text{sat}}_M$, this result implies that $\dot{\gamma}(\dot{x}, t)$ is also bounded. As a consequence, it follows from (25) that u, ω_1 , and ω_2 are well-defined and bounded along every system's solution. This and the boundedness of ω_3 yield Property 1 of the theorem. Let us now establish Property 3. Omitting the arguments for γ and $\gamma_{e,d}$, we have that

$$\dot{v} = -S(\omega)\tilde{v} - ue_3 + R^T \gamma + R^T (\gamma_{e,d} - \overline{\text{sat}}_M(\gamma_{e,d}))$$

Therefore, the time-derivative of the function V defined by (12) along the system's solutions is given by

$$\begin{aligned} \dot{V} &= \tilde{v}_3 (-u + \bar{\gamma}_3) + \tilde{v}^T R^T (\gamma_{e,d} - \overline{\text{sat}}_M(\gamma_{e,d})) \\ &+ \frac{1}{|\gamma|k_2} \bar{\gamma}_{1,2}^T \left[\begin{pmatrix} -\omega_2 \\ \omega_1 \end{pmatrix} + \frac{1}{|\gamma|^2} \begin{pmatrix} -\gamma^T S(Re_2) \dot{\gamma} \\ \gamma^T S(Re_1) \dot{\gamma} \end{pmatrix} + |\gamma|k_2 \tilde{v}_{1,2} \right] \end{aligned}$$

Replacing u, ω_1, ω_2 by their expressions in (25) one obtains

$$\begin{aligned} \dot{V} &= -|\gamma|k_1 \tilde{v}_3^2 - \mu_\tau (|\gamma| + \bar{\gamma}_3) \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2} \\ &+ \tilde{v}^T R^T (\gamma_{e,d} - \overline{\text{sat}}_M(\gamma_{e,d})) \\ &+ \frac{(1 - \mu_\tau (|\gamma|))}{|\gamma|^3 k_2} (-\bar{\gamma}_1 \gamma^T S(Re_2) + \bar{\gamma}_2 \gamma^T S(Re_1)) \dot{\gamma} \end{aligned} \quad (51)$$

It follows from (30) and Assumption 4 that $\overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}_r(t), t)) = \gamma_{e,d}(\dot{x}_r(t), t)$, when $M \geq \bar{c}_1 + \bar{c}_2 \bar{\mathbf{v}}_r^2$. Using Assumption 2 one deduces that $\overline{\text{sat}}_M(\gamma_{e,d}(\dot{x}, t)) = \gamma_{e,d}(\dot{x}, t)$ for \dot{x} in a neighborhood of \dot{x}_r , when $M > \bar{c}_1 + \bar{c}_2 \bar{\mathbf{v}}_r^2$. Furthermore, since $\tau \in (0, \delta)$ by assumption, one deduces from (24) that $\mu_\tau (|\gamma|) = 1$ in a neighborhood of \dot{x}_r . Equation (51) becomes

$$\dot{V} = -|\gamma|k_1 \tilde{v}_3^2 - \mu_\tau (|\gamma| + \bar{\gamma}_3) \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2}$$

and the proof of local asymptotic stability proceeds like the proof of Proposition 2. Let us now consider the case when $M \geq \bar{c}_1 + \bar{c}_2(\kappa(\bar{c}_i, \bar{\mathbf{v}}_r))^2$ and $|\gamma(\dot{x}, t)| \geq \tau, \forall(\dot{x}, t)$. This latter condition implies that $\mu_\tau (|\gamma|) = 1, \forall(\dot{x}, t)$. Therefore, equation (51) becomes

$$\begin{aligned} \dot{V} &= -|\gamma|k_1 \tilde{v}_3^2 - \mu_\tau (|\gamma| + \bar{\gamma}_3) \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2} \\ &+ \tilde{v}^T R^T (\gamma_{e,d} - \overline{\text{sat}}_M(\gamma_{e,d})) \\ &\leq -|\gamma|k_1 \tilde{v}_3^2 - \mu_\tau (|\gamma| + \bar{\gamma}_3) \frac{k_3}{k_2} \frac{|\bar{\gamma}_{1,2}|^2}{(|\gamma| + \bar{\gamma}_3)^2} \end{aligned}$$

where the inequality follows from Lemma 3. From here the proof proceeds like the proof of Proposition 2.

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