

Transverse functions on special orthogonal groups for vector fields satisfying the LARC at the order one

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Abstract—The transverse function approach, a control design method initially developed by the authors for nonlinear driftless systems, is based on a theorem that establishes the equivalence between the satisfaction of the Lie algebra rank condition (LARC) by a family of vector fields and the existence of functions "transverse" to these vector fields, defined on a torus of adequate dimension. In this paper we prove the existence of transverse functions defined on special orthogonal groups for vector fields which satisfy the LARC at the order one, and we provide an explicit expression of such functions. An example illustrates some of the advantages resulting from the use of these new functions.

I. INTRODUCTION

The transverse function (TF) approach [1], [2] is a control design method developed for so-called "critical" nonlinear control systems, i.e., controllable systems with non-controllable linear approximation at equilibrium points. It was originally developed for the control of driftless systems of the form

$$\dot{q} = \sum_{i=1}^m X_i(q)u_i \quad (1)$$

with X_1, \dots, X_m smooth vector fields (v.f.) on some manifold M , q the state, and $u = (u_1, \dots, u_m)'$ the control vector. These systems are critical when $m < n := \dim(q)$. Kinematic models of nonholonomic mechanical systems (wheeled robots, rolling spheres, etc) belong to this class of systems, and the TF approach has been applied to several of them [3], [4], [5], [6], [7]. More recently, it has also been applied to critical underactuated mechanical systems (i.e. systems with fewer force/torque control inputs than degrees of freedom) [8], [9]. With respect to classical feedback design methods aiming at the *asymptotic* stabilization of zero for some tracking error, the primary objective of the TF approach is to ensure *practical* stabilization, i.e. stabilization of a neighborhood of zero. This allows for new control solutions endowed with new properties (see [10] for more details), such as the capability of ensuring the stabilization of *any* reference trajectory, i.e. not necessarily admissible. In addition, the ultimate tracking error can be rendered arbitrarily small by a proper choice of the control parameters. This result comes as an interesting counterpart to the non-existence of a feedback controller capable of asymptotically stabilizing any admissible reference trajectory [11].

The TF approach is based on a theorem proved in [1] and further refined in [2], that shows the equivalence between the following properties of a family of smooth vector fields X_1, \dots, X_m on a manifold M : *i*) this family of v.f. satisfies the Lie Algebra Rank Condition (LARC) at some point q_0 , *ii*) there exist functions $f : \mathbb{T}^p \rightarrow \mathcal{U}(q_0) \subset M$ transverse to X_1, \dots, X_m in the sense that for any $\theta \in \mathbb{T}^p$, $X_1(f(\theta)), \dots, X_m(f(\theta)), \frac{\partial f}{\partial \theta_1}(\theta), \dots, \frac{\partial f}{\partial \theta_p}(\theta)$ span the tangent space of M at $f(\theta)$. Here \mathbb{T}^p denotes the p -dimensional torus and $\mathcal{U}(q_0)$ is a neighborhood of q_0 . Given a controllable control system (1), the existence of "transverse functions" for this system is ascertained by the above-mentioned theorem. The design of practical stabilizers based on the TF approach relies on the explicit determination of such functions (see [2] for details). In [2], expressions of TF are given. Exploring other possibilities is important because the performance of the controlled system is related to the properties of the TF. Efforts have already been devoted to the determination of new TF with complementary properties, e.g. to achieve the asymptotic stabilization of feasible trajectories [10], but this topic is still very open.

In this paper we depart from the original formulation of the TF approach by introducing TF defined on special orthogonal groups $\mathbb{SO}(m)$ instead of tori \mathbb{T}^p . Given a family of v.f. $\{X_1, \dots, X_m\}$ such that the v.f. $X_i, [X_i, X_j]$ $i, j = 1 \dots, m$ span the tangent space of M at some point q_0 , we prove the existence of TF defined on $\mathbb{SO}(m)$ and provide explicit expressions for these functions. A noticeable feature of these functions is that they present symmetry properties respectful of those of the associated control system. Such functions have recently been used for the control of the "trident-snake" mechanism (a snake-like robot introduced by M. Ishikawa in [12]), with improved performance and control efficiency. This application is presented in a separate paper [13]. It is probable that these new TF can also be used for other mechanical systems. The present study also initiates a long term research about the characterization of manifolds on which TF can be defined.

The paper is organized as follows. Section II presents the notation and recalls basic definitions associated with the transverse function approach. An example is addressed in Section III in order to motivate the introduction of the new

TF and provide comparisons with respect to (w.r.t.) classical ones. The main results are presented in Section IV. Proofs are given in the appendix.

II. NOTATION AND RECALLS

A. Vectors and manifolds

The transpose of a vector $x \in \mathbb{R}^n$ is denoted as x' , its i -th component as x_i , and its Euclidean norm as $|x|$. The i -th vector of the canonical basis of \mathbb{R}^n is denoted as e_i , i.e. $e_i'x = x_i$. The $m \times m$ identity matrix is denoted as I_m . The tangent space at q of a manifold M is denoted as T_qM . Given a family $X^1 = \{X_1, \dots, X_m\}$ of smooth v.f. on M and a vector $\xi \in \mathbb{R}^m$, we denote by $X^1(q)\xi$ the tangent vector $\sum_{i=1}^m X_i(q)\xi_i \in T_qM$. The special orthogonal group $\mathbb{SO}(m)$ is the matrix Lie group $\{R \in \mathbb{R}^{m \times m} : RR' = I_m, \det(R) = 1\}$. For any smooth curve $R(\cdot)$ on $\mathbb{SO}(m)$ and any basis $S_1, \dots, S_{m(m-1)/2}$ of $m \times m$ skew-symmetric matrices (i.e. any basis of the Lie algebra $\mathfrak{so}(m)$ of $\mathbb{SO}(m)$), one has $\dot{R} = RS(\omega)$ with $S(\omega) = \sum_i \omega_i S_i$, for some time functions $\omega_i(\cdot)$. Finally, given a family $X^1 = \{X_1, \dots, X_m\}$ of smooth v.f. on a manifold M , we say that X^1 satisfies the LARC at "the order one" at some point q_0 if

$$\text{span}\{X_i(q_0), [X_i, X_j](q_0), i, j = 1, \dots, m\} = T_{q_0}M \quad (2)$$

B. Exterior algebra and wedge product

Standard notation and definitions about exterior algebra and wedge product are now recalled (see e.g. [14] or [15] for more details). The *exterior algebra* $\bigwedge V$ of a m -dimensional vector space V over \mathbb{R} is a unital associative algebra generated by the *wedge product* \wedge . The wedge product operation $\wedge : (\bigwedge V) \times (\bigwedge V) \rightarrow \bigwedge V$ is associative, bilinear, and alternating in the sense that $v \wedge v = 0$ for all $v \in V$. Note that this is equivalent to the property $v \wedge w = -w \wedge v$, $\forall (v, w) \in V^2$. As a vector space, $\bigwedge V$ can be decomposed as $\bigwedge V = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \dots \oplus \bigwedge^m V$, with $\bigwedge^0 V = \mathbb{R}$, $\bigwedge^1 V = V$, and each $\bigwedge^r V$ ($r \geq 1$) the vector space generated by elements of the form $v_1 \wedge \dots \wedge v_r$, with $v_1, \dots, v_r \in V$. One deduces from the alternating property of the wedge product that given any basis $\{e_1, \dots, e_m\}$ of V , and any r , the set

$$\{e_{i_1} \wedge \dots \wedge e_{i_r} : i_1 < \dots < i_r\} \quad (3)$$

defines a basis of $\bigwedge^r V$.

Given a linear function $R : V \rightarrow V$, a *homomorphic extension* of R to $\bigwedge^r V$ is a linear function $R_r : \bigwedge^r V \rightarrow \bigwedge^r V$ such that

$$R_r(v_1 \wedge \dots \wedge v_r) = (Rv_1) \wedge \dots \wedge (Rv_r), \quad \forall (v_1, \dots, v_r) \in V^r \quad (4)$$

From [14, Th. 2.19.1], any linear function R can be associated (for each r) with one and only one homomorphic extension. We will use the following property, the proof of which results from the fact that (3) is a basis of $\bigwedge^r V$ whenever $\{e_1, \dots, e_m\}$ is a basis of V .

Lemma 1 *If a linear function $R : V \rightarrow V$ is bijective, then for any r its homomorphic extension R_r is also bijective.*

C. Systems on Lie groups

Let G denote a connected Lie group of dimension n . The unit element of G is denoted as e , i.e. $\forall g \in G : ge = eg = g$. The inverse g^{-1} of $g \in G$ is the (unique) element in G such that $gg^{-1} = g^{-1}g = e$. The left (resp. right) translation operator on G is denoted as L (resp. R), i.e. $\forall (\sigma, \tau) \in G^2 : L_\sigma(\tau) = R_\tau(\sigma) = \sigma\tau$. A v.f. X on G is left-invariant iff $\forall (\sigma, \tau) \in G^2, dL_\sigma(\tau)X(\tau) = X(\sigma\tau)$, with df denoting the differential of a function f . The Lie algebra –of left-invariant v.f.– of the group G is denoted as \mathfrak{g} . If $X \in \mathfrak{g}$, $\exp(tX)$ is the solution at time t of $\dot{g} = X(g)$ with the initial condition $g(0) = e$. A driftless control system $\dot{g} = \sum_{i=1}^m X_i(g)\xi_i$ on G is said to be left-invariant on G if the control v.f. X_i are left-invariant.

D. Transverse Functions

Definition and general characterization Let $X^1 = \{X_1, \dots, X_m\}$ denote a family of smooth v.f. X_1, \dots, X_m on a n -dimensional manifold M and \mathbb{T}^p denote the p -dimensional torus. A smooth function $f : \mathbb{T}^p \rightarrow M$ is *transverse to X^1* if, for any $\theta \in \mathbb{T}^p$,

$$\text{span}\{X_1(f(\theta)), \dots, X_m(f(\theta)), df(\theta)(T_\theta\mathbb{T}^p)\} = T_{f(\theta)}M$$

with df the differential of f . Note that p , the dimension of \mathbb{T}^p , must be at least equal to $(n-m)$. Given smooth functions $f^\varepsilon : \mathbb{T}^p \rightarrow M$ defined for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$, we say that (f^ε) is a *family of functions transverse to X^1* if $\forall \varepsilon \in (0, \varepsilon_0)$, f^ε is transverse to X^1 . Given $q_0 \in M$ such that the family X^1 satisfies the LARC at q_0 , the "transverse function theorem" given in [1] ensures the existence of a family of functions transverse to X^1 , with $\max_\theta \text{dist}(f^\varepsilon(\theta), q_0) \rightarrow 0$ as $\varepsilon \rightarrow 0$, where "dist" denotes any distance locally defined in the neighborhood of q_0 .

The case of invariant v.f. on Lie groups When $M = G$ is a Lie group and X_1, \dots, X_m are independent elements¹ of the Lie algebra \mathfrak{g} , stronger results can be obtained (see [2] for details). First, provided that the family X^1 satisfies the LARC at e , functions transverse to X^1 can be defined on \mathbb{T}^{n-m} , i.e. with the minimal value $(n-m)$ of p . A family (f^ε) of such functions, with the property that $\max_\theta \text{dist}(f^\varepsilon(\theta), e) \rightarrow 0$ as $\varepsilon \rightarrow 0$, is given by

$$f^\varepsilon(\theta) = f_n^\varepsilon(\theta_n) f_{n-1}^\varepsilon(\theta_{n-1}) \dots f_{m+1}^\varepsilon(\theta_{m+1}) \quad (5)$$

with each f_i^ε defined by

$$f_i^\varepsilon(\theta_i) = \exp((\varepsilon a_i)^{\alpha_i} \sin \theta_i X_{\lambda(i)} + (\varepsilon b_i)^{\beta_i} \cos \theta_i X_{\rho(i)}) \quad (6)$$

for some constant scalars $a_i, \alpha_i, b_i, \beta_i$ and some v.f. $X_{\lambda(i)}, X_{\rho(i)} \in \mathfrak{g}$. Given functions f^ε transverse to X^1 , one can design feedback controls that practically stabilize any reference trajectory q_r for the associated control system (1). The idea is to asymptotically stabilize $z = q_r^{-1}q(f^\varepsilon(\theta))^{-1}$ at e (an easy task due to the transversality property), which implies the convergence of q to $q_r f^\varepsilon(\theta)$. Since $\max_\theta \text{dist}(f^\varepsilon(\theta), e) \rightarrow 0$ as $\varepsilon \rightarrow 0$, the ultimate

¹a property equivalent to $X_1(e), \dots, X_m(e)$ being independent.

distance between q and q_r can be rendered arbitrarily small via the choice of ε . Note the importance of defining f^ε on a *compact* manifold, in order to ensure boundedness of the tracking error. This is one of the constraints that complicate the design of TF. On the other hand, this manifold does not have to be a torus.

III. A MOTIVATING EXAMPLE

Let us consider the driftless system

$$\begin{cases} \dot{x} &= u \\ \dot{y} &= x \times u \end{cases} \quad (7)$$

with $x, y, u \in \mathbb{R}^3$ and \times the cross product. This system can be used as an homogeneous approximation of the trident-snake robot kinematics [16]. It is of the form (1) with $m = 3$, $q = (x, y)$ and $X_i(q) = (e_i, x \times e_i); i = 1, 2, 3$. One easily verifies that the family $X^1 = \{X_1, X_2, X_3\}$ satisfies the LARC at the order one at any q_0 . In addition, (7) is a system on a Lie group with the group product between $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ defined as

$$q_1 q_2 = (x_1 + x_2, y_1 + y_2 + x_1 \times x_2) \quad (8)$$

and the unit element $e = (0, 0)$. Functions transverse to X^1 can be defined according to (5), i.e.

$$f^\varepsilon(\theta) = f_6^\varepsilon(\theta_6) f_5^\varepsilon(\theta_5) f_4^\varepsilon(\theta_4) \quad (9)$$

with

$$\begin{aligned} f_4^\varepsilon(\theta_4) &= \exp(\varepsilon a_4 \sin \theta_4 X_1 + \varepsilon b_4 \cos \theta_4 X_2) \\ f_5^\varepsilon(\theta_5) &= \exp(\varepsilon a_5 \sin \theta_5 X_1 + \varepsilon b_5 \cos \theta_5 X_3) \\ f_6^\varepsilon(\theta_6) &= \exp(\varepsilon a_6 \sin \theta_6 X_2 + \varepsilon b_6 \cos \theta_6 X_3) \end{aligned}$$

There remains to determine an analytical expression of the function f^ε and find conditions on the a_i 's and b_i 's that ensure the property of transversality, i.e. the invertibility of the square matrix

$$\begin{pmatrix} X_1(f^\varepsilon(\theta)) & X_2(f^\varepsilon(\theta)) & X_3(f^\varepsilon(\theta)) & \frac{\partial f^\varepsilon}{\partial \theta_4}(\theta) & \frac{\partial f^\varepsilon}{\partial \theta_5}(\theta) & \frac{\partial f^\varepsilon}{\partial \theta_6}(\theta) \end{pmatrix}$$

for any θ . Instead, we show next how another family of TF can be obtained.

Let $R(\cdot)$ denote an arbitrary smooth curve on the special orthogonal group $\mathbb{SO}(3)$, and ω the associated "velocity vector", i.e. such that $\dot{R} = RS(\omega)$ with $S(\omega)$ the skew-symmetric matrix associated with the cross product, i.e. such that $S(\omega)x = \omega \times x, \forall x \in \mathbb{R}^3$. Consider the set of functions $f^\varepsilon : \mathbb{SO}(3) \rightarrow \mathbb{R}^6$ defined by

$$f^\varepsilon(R) = (f_x^\varepsilon(R), f_y^\varepsilon(R)) = (\varepsilon Ra, \varepsilon^2 Rb) \quad (10)$$

with $a, b \in \mathbb{R}^3$ some constant vectors. Note that $\max_R d(f^\varepsilon(R), e) \rightarrow 0$ as $\varepsilon \rightarrow 0$. By a direct extension of the definition of a TF, we say that f^ε is transverse to X^1 if, for any $R \in \mathbb{SO}(3)$,

$$\text{span}\{X_1(f^\varepsilon(R)), \dots, X_3(f^\varepsilon(R)), df^\varepsilon(R)(T_R \mathbb{SO}(3))\} = \mathbb{R}^6$$

Along any smooth curve $R(\cdot)$, the function f^ε defined by (10) satisfies

$$\begin{aligned} \dot{f}^\varepsilon(R) &= (\varepsilon RS(\omega)a, \varepsilon^2 RS(\omega)b) \\ &= -(\varepsilon RS(a)\omega, \varepsilon^2 RS(b)\omega) \end{aligned} \quad (11)$$

Then, it follows from the definition of transversality and (7) that f^ε is a TF iff the square matrix

$$\begin{pmatrix} I_3 & \varepsilon RS(a) \\ S(\varepsilon Ra) & \varepsilon^2 RS(b) \end{pmatrix}$$

is invertible for any R . Using the fact that $S(Rx) = RS(x)R'$ for any x , one easily verifies that this property is satisfied iff $\varepsilon \neq 0$ and $S(b) - S(a)^2$ is invertible, i.e. iff $\varepsilon \neq 0$ and $a \times b \neq 0$.

This example illustrates the possibility to define TF on $\mathbb{SO}(3)$, with simple explicit transversality conditions bearing upon the choice of the parameters a and b in (10). In addition, these TF respect the "symmetry" (see e.g. [17] for more details on this topic) of System (7) w.r.t. rotations involved in changes of coordinates. More precisely, given a constant rotation matrix R_0 and a solution (x, y) to (7) associated with an input u , then $(R_0 x, R_0 y)$ is also a solution associated with the input $R_0 u$. Similarly, for any curve $f^\varepsilon(R(\cdot)) = (f_x^\varepsilon(R(\cdot)), f_y^\varepsilon(R(\cdot)))$ on $f^\varepsilon(\mathbb{SO}(3))$, then $(R_0 f_x^\varepsilon(R(\cdot)), R_0 f_y^\varepsilon(R(\cdot)))$ is also a curve on $f^\varepsilon(\mathbb{SO}(3))$ since $(R_0 f_x^\varepsilon(R), R_0 f_y^\varepsilon(R)) = f^\varepsilon(R_0 R)$. This type of symmetry is not granted by the TF (9). In particular, the TF obtained by interchanging the order of the products in the right-hand side of (9) are all different, due to the skew-symmetry of the cross product. This lack of symmetry in turn induces more stringent conditions on the functions' parameters in order to ensure the property of transversality.

IV. MAIN RESULTS

Let $X^1 = \{X_1, \dots, X_m\}$ denote a family of smooth v.f. on a manifold M . Throughout this section, we assume that X^1 satisfies the LARC at "the order one" at some point q_0 (see Eq. (2)). Based on this assumption we show the existence of functions transverse to X^1 defined on the special orthogonal group $\mathbb{SO}(m)$ and provide explicit expressions of such functions. This is first accomplished for a "canonical system" that generalizes system (7).

Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{m(m-1)/2}$. Since $m(m-1)/2 = \dim(\wedge^2 \mathbb{R}^m)$ and since a basis of $\wedge^2 \mathbb{R}^m$ is

$$\{e_i \wedge e_j : 1 \leq i < j \leq m\}$$

we can identify y with an element of $\wedge^2 \mathbb{R}^m$, i.e.

$$y = \sum_{1 \leq i < j \leq m} y_{ij} e_i \wedge e_j \quad (12)$$

Based on this identification, let us consider the system

$$\begin{cases} \dot{x} &= u \\ \dot{y} &= x \wedge u \end{cases} \quad (13)$$

From (12), the second equation of (13) can be rewritten as

$$\sum_{1 \leq i < j \leq m} \dot{y}_{ij} e_i \wedge e_j = \left(\sum_{i=1}^m x_i e_i \right) \wedge \left(\sum_{j=1}^m u_j e_j \right)$$

from which we deduce, by anticommutativity of the wedge product,

$$\dot{y}_{ij} = x_i u_j - x_j u_i, \quad 1 \leq i < j \leq m \quad (14)$$

This shows how System (13) is a generalization of (7). It is also of the form (1), with

$$X_i(q) = (e_i, \sum_{k=1}^m x_k e_k \wedge e_i), \quad i = 1, \dots, m \quad (15)$$

One easily verifies that the controllability condition (2) is satisfied at any point. In particular, $[X_i, X_j](q) = (0, 2e_i \wedge e_j)$, $\forall i, j, \forall q$. One can also verify that (13) is a system on a Lie group, with the group product between $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ defined by

$$q_1 q_2 = (x_1 + x_2, y_1 + y_2 + x_1 \wedge x_2) \quad (16)$$

and the unit element $e = (0, 0)$.

Theorem 1 Consider the v.f. X_1, \dots, X_m defined by (15). Let

$$\alpha^\varepsilon(R) = \varepsilon R a_1, \quad \beta^\varepsilon(R) = \varepsilon^2 \sum_{k=1}^{m-2} (R a_k) \wedge (R a_{k+1}) \quad (17)$$

with $R \in \mathbb{S}\mathbb{O}(m)$ and $\{a_1, \dots, a_{m-1}\}$ a set of non-zero orthogonal vectors in \mathbb{R}^m , i.e. such that $a_i' a_j = 0, \forall (i, j)$.

Then, for any $\varepsilon \neq 0$, the function $f^\varepsilon : \mathbb{S}\mathbb{O}(m) \rightarrow \mathbb{R}^n$ defined by

$$f^\varepsilon(R) = (\alpha^\varepsilon(R), \beta^\varepsilon(R)) \quad (18)$$

is transverse to $\{X_1, \dots, X_m\}$.

Examples: When $m = 3$, the wedge product is isomorphic to the cross product so that, from (17),

$$\beta^\varepsilon(R) = \varepsilon^2 (R a_1) \times (R a_2) = \varepsilon^2 R (a_1 \times a_2)$$

The function f^ε defined by (18) is thus given by

$$f^\varepsilon(R) = (\varepsilon R a_1, \varepsilon^2 R (a_1 \times a_2))$$

This is the same function as in (10), with $b = a_1 \times a_2$.

When $m = 2$, any element $R \in \mathbb{S}\mathbb{O}(2)$ can be identified with the rotation matrix of some angle θ , i.e.

$$R = R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Let $a_1 = e_1$. Since $\beta^\varepsilon(R) = 0$ when $m = 2$, (18) yields

$$f^\varepsilon(R) = (\varepsilon R e_1, 0) = \left(\begin{pmatrix} \varepsilon \cos \theta \\ \varepsilon \sin \theta \end{pmatrix}, 0 \right)$$

This function coincides with the transverse function (6) defined on \mathbb{T} , with the specific choice $a_i = b_i = \alpha_i = \beta_i = 1$, $X_\lambda(i) = X_2$, and $X_\rho(i) = X_1$.

A generalization of Theorem 1 to any family of left-invariant v.f. on a Lie group that satisfies (2) is stated next.

Theorem 2 Let X_1, \dots, X_m denote left-invariant vector fields on a Lie group G , that satisfy the LARC at the order one at the unit element e of G . Denote α_i^ε , $i = 1, \dots, m$ the coordinates of α^ε in the basis $\{e_i : i = 1, \dots, m\}$, and β_{ij}^ε , $1 \leq i < j \leq m$ the coordinates of β^ε in the basis $\{e_i \wedge e_j : 1 \leq i < j \leq m\}$, with α^ε and β^ε defined by (17).

Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ and any $q_0 \in G$, the function $f^\varepsilon : \mathbb{S}\mathbb{O}(m) \rightarrow G$ defined by

$$f^\varepsilon(R) = \exp \left(\sum_{i=1}^m \alpha_i^\varepsilon(R) X_i + \frac{1}{2} \sum_{1 \leq i < j \leq m} \beta_{ij}^\varepsilon(R) [X_i, X_j] \right) (q_0) \quad (19)$$

is transverse to $\{X_1, \dots, X_m\}$, with $\exp(Y)(q_0)$ denoting the solution at time $t = 1$ of the system $\dot{q} = Y(q)$ with initial condition $q(0) = q_0$.

Equation (19) associates a family of TF with an arbitrary point q_0 . These functions can also be deduced, by left translation, from a family of TF associated with the unit element $q_0 = e$, using the fact that for any left-invariant v.f. Y , $\exp(Y)(q_0) = q_0 \exp(Y)(e)$.

The determination of the function f^ε in (19) requires to integrate a differential equation. For most nonlinear v.f., this integration is complex and does not yield a closed-form expression. To circumvent this difficulty, one can work with local coordinates and replace each X_i in Formula (19) by an homogeneous approximation Y_i (see [1] for more details on this issue, and [18] for a survey on homogeneous approximations of vector fields). Since these homogeneous approximations Y_1, \dots, Y_m generate a nilpotent Lie algebra, the exp mapping on this Lie algebra reduces to a finite expansion and f^ε can be calculated with a finite number of algebraic operations.

When $M = \mathbb{R}^n$ or when the v.f. X_i are expressed in local coordinates, one can also use an approximation of the function (19) obtained by expansion of the exponential map. More precisely, one can show that

$$\bar{f}^\varepsilon(R) = q_0 + L_Z(\text{id})(q_0) + \frac{1}{2} L_Z(L_Z(\text{id}))(q_0)$$

also defines a TF for ε small enough, with

$$Z = \sum_{i=1}^m \alpha_i^\varepsilon(R) X_i + \frac{1}{2} \sum_{1 \leq i < j \leq m} \beta_{ij}^\varepsilon(R) [X_i, X_j]$$

id the identity function on \mathbb{R}^n and, for any smooth function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$L_Z(\varphi)(q_0) = \begin{pmatrix} L_Z(\varphi_1)(q_0) \\ \vdots \\ L_Z(\varphi_n)(q_0) \end{pmatrix}$$

with $L_Z(\varphi_k)$ the Lie derivative of φ_k along the v.f. Z , i.e. $L_Z(\varphi_k)(q_0) = \frac{\partial \varphi_k}{\partial x} (q_0) Z(q_0)$.

Finally, let us remark that Theorem 2 can itself be generalized to any family of v.f. on a manifold (i.e. not necessarily a Lie group) that satisfies (2) at a point q_0 . In particular, Formula (19) is still valid in this case and the above remarks concerning the approximation of the function f^ε , either by approximation of the v.f. X_i or by approximation of the exp mapping, also apply.

CONCLUSION

We have derived TF defined on special orthogonal groups for families of v.f. that satisfy a first-order controllability condition. Possible extensions to this study are several. One of them concerns the extension of this result to systems for which higher-order Lie brackets are necessary to generate all directions of the tangent space. Another issue concerns the characterization of manifolds that can be used as domains of TF. The use of these functions to control various nonholonomic and/or underactuated mechanical systems also offers a large domain of investigation.

APPENDIX

Proof of Theorem 1:

By definition, the property of transversality is satisfied if, for any $R \in \mathbb{S}\mathbb{O}(m)$, the mapping

$$\begin{pmatrix} u \\ \omega \end{pmatrix} \mapsto \sum_{i=1}^m u_i X_i(f^\varepsilon(R)) + \dot{f}^\varepsilon(R) = \begin{pmatrix} u + \dot{\alpha}^\varepsilon(R) \\ \alpha^\varepsilon(R) \wedge u + \beta^\varepsilon(R) \end{pmatrix}$$

is onto, with ω related to \dot{R} by the relation $\dot{R} = RS(\omega)$ (see Section II-A for the notation). One verifies that this is equivalent to the property of the mapping $h_R^\varepsilon : \mathbb{R}^{m(m-1)/2} \rightarrow \bigwedge^2 \mathbb{R}^m$ defined by

$$\omega \mapsto h_R^\varepsilon(\omega) = \beta^\varepsilon(R) - \alpha^\varepsilon(R) \wedge \dot{\alpha}^\varepsilon(R) \quad (20)$$

of being onto. It follows from (17) that $h_R^\varepsilon = \varepsilon^2 h_R^1$ so that it is sufficient to prove that h_R^1 is onto. Furthermore, since h_R^1 is linear and $\dim(\omega) = \dim(h_R^1(\omega)) = m(m-1)/2$, this property is itself equivalent to the injectivity of h_R^1 . From (17),

$$\begin{aligned} h_R^1(\omega) &= \sum_{k=1}^{m-2} (\dot{R}a_k) \wedge (Ra_{k+1}) + \sum_{k=1}^{m-2} (Ra_k) \wedge (\dot{R}a_{k+1}) \\ &\quad - Ra_1 \wedge \dot{R}a_1 \\ &= \sum_{k=1}^{m-2} (RS(\omega)a_k) \wedge (Ra_{k+1}) \\ &\quad + \sum_{k=1}^{m-2} (Ra_k) \wedge (RS(\omega)a_{k+1}) - Ra_1 \wedge RS(\omega)a_1 \\ &= R_2 \left(\sum_{k=1}^{m-2} (S(\omega)a_k) \wedge a_{k+1} \right. \\ &\quad \left. + \sum_{k=1}^{m-2} a_k \wedge (S(\omega)a_{k+1}) - a_1 \wedge S(\omega)a_1 \right) \end{aligned}$$

with R_2 the homomorphic extension of R on $\bigwedge^2 \mathbb{R}^m$, as defined by (4). It follows from Lemma 1 that the property of transversality is satisfied when $\varepsilon \neq 0$ iff the mapping h_R^1 defined by

$$\begin{aligned} h_R^1(\omega) &= \sum_{k=1}^{m-2} ((S(\omega)a_k) \wedge a_{k+1} + a_k \wedge (S(\omega)a_{k+1})) \\ &\quad - a_1 \wedge S(\omega)a_1 \end{aligned}$$

is injective.

By assumption $\{a_1, \dots, a_{m-1}\}$ are non-zero orthogonal vectors in \mathbb{R}^m . Therefore, there exists a_m so that

$\{a_1, \dots, a_m\}$ forms an orthogonal basis of \mathbb{R}^m . We parameterize the matrix $S(\omega)$ as follows:

$$S(\omega) = \sum_{1 \leq i < j \leq m} \omega_{i,j} S_{i,j}, \quad S_{i,j} = (a_i a_j' - a_j a_i')$$

One easily verifies that the matrices $S_{i,j}$ form a basis of $\mathfrak{so}(m)$. Using this parameterization, one obtains the following decomposition of $h_R^1(\omega)$ as a linear combination of elements $a_i \wedge a_j$ ($i < j$):

$$\begin{aligned} h_R^1(\omega) &= \sum_{1 < j} \lambda_1 \omega_{1,j} a_1 \wedge a_j \\ &\quad - \sum_{j=2}^{m-2} \sum_{i < j} \lambda_{j+1} \omega_{i,j+1} a_i \wedge a_j + \sum_{j=2}^{m-1} \sum_{i < j-1} \lambda_{j-1} \omega_{i,j-1} a_i \wedge a_j \\ &\quad - \sum_{i=1}^{m-2} \sum_{i+1 < j} \lambda_{i+1} \omega_{i+1,j} a_i \wedge a_j + \sum_{i=2}^{m-1} \sum_{i < j} \lambda_{i-1} \omega_{i-1,j} a_i \wedge a_j \end{aligned}$$

with $\lambda_i = |a_i|^2$. This expression in turn allows one to determine the coordinates $c_{ij}(\omega)$ of $h_R^1(\omega)$ in the basis $\{a_i \wedge a_j : i < j\}$ of $\bigwedge^2 \mathbb{R}^m$, i.e.

$$h_R^1(\omega) = \sum_{i < j} c_{ij}(\omega) a_i \wedge a_j \quad (21)$$

In particular, one has

$$\begin{aligned} &\sum_{j=2}^m c_{1,j}(\omega) a_1 \wedge a_j = \\ &\lambda_1 \sum_{1 < j} \omega_{1,j} a_1 \wedge a_j - \lambda_2 \sum_{2 < j} \omega_{2,j} a_1 \wedge a_j \\ &\quad - \sum_{j=2}^{m-2} \lambda_{j+1} \omega_{1,j+1} a_1 \wedge a_j + \sum_{j=3}^{m-1} \lambda_{j-1} \omega_{1,j-1} a_1 \wedge a_j \end{aligned} \quad (22)$$

and, for any $i > 1$,

$$\begin{aligned} &\sum_{j=i+1}^m c_{i,j}(\omega) a_i \wedge a_j = \\ &\lambda_{i-1} \sum_{i < j} \omega_{i-1,j} a_i \wedge a_j - \lambda_{i+1} \sum_{i+1 < j} \omega_{i+1,j} a_i \wedge a_j \\ &\quad - \sum_{j=i+1}^{m-2} \lambda_{j+1} \omega_{i,j+1} a_i \wedge a_j + \sum_{j=i+2}^{m-1} \lambda_{j-1} \omega_{i,j-1} a_i \wedge a_j \end{aligned} \quad (23)$$

From (22) one gets

$$\sum_{j=2}^m \lambda_j \omega_{1,j} c_{1,j}(\omega) = \lambda_1 \sum_{1 < j} \lambda_j \omega_{1,j}^2 - \lambda_2 \sum_{2 < j} \lambda_j \omega_{1,j} \omega_{2,j} \quad (24)$$

and from (23) one gets, for any $i > 1$,

$$\begin{aligned} &\sum_{j=i+1}^m \lambda_j \omega_{i,j} c_{i,j}(\omega) = \\ &\lambda_{i-1} \sum_{i < j} \lambda_j \omega_{i,j} \omega_{i-1,j} - \lambda_{i+1} \sum_{i+1 < j} \lambda_j \omega_{i,j} \omega_{i+1,j} \end{aligned} \quad (25)$$

It follows from (24) and (25) that

$$\sum_{i=1}^{m-1} \lambda_i \sum_{j=i+1}^m \lambda_j \omega_{i,j} c_{i,j}(\omega) = \lambda_1^2 \sum_{1 < j} \lambda_j \omega_{1,j}^2 \quad (26)$$

The injectivity of h_I^1 easily follows from the above relations. Indeed, suppose that $h_I^1(\omega) = 0$ and let us show that $\omega = 0$. We proceed by induction. By (21), $c_{i,j}(\omega) = 0$ for any (i, j) so that, by (26), $\omega_{1,j} = 0 \forall j$, since all λ_i 's are strictly positive numbers. Now assume that $\omega_{k,j} = 0$ for any $k = 1, \dots, i$ and any $j > k$, then $\omega_{i+1,j} = 0$ for any $j > i + 1$. This follows from (22) when $i = 1$, and from (23) when $i > 1$, using the fact that all $c_{i,j}$'s are equal to zero. ■

Proof of Theorem 2:

Recall the classical formula [19, Pg. 105] for the derivative of the exponential mapping on Lie groups:

$$\frac{d}{ds} \exp(X + sY)|_{s=0} = (\phi(\text{ad}X), Y)(\exp X) \quad (27)$$

with

$$\phi(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} z^j$$

Consider any smooth curve $t \mapsto R(t)$ on $\mathbb{S}\mathbb{O}(m)$. Then, using (27), one obtains the following expression for $\dot{f}^\varepsilon := df^\varepsilon(R(t))/dt$:

$$\dot{f}^\varepsilon = (\phi(\text{ad}Z_1), Z_2)(f^\varepsilon) \quad (28)$$

with

$$Z_1 = \sum_{i=1}^m \alpha_i^\varepsilon(R) X_i + \sum_{i < j} \beta_{ij}^\varepsilon(R) [X_i, X_j]$$

$$Z_2 = \sum_{i=1}^m \dot{\alpha}_i^\varepsilon(R) X_i + \sum_{i < j} \dot{\beta}_{ij}^\varepsilon(R) [X_i, X_j]$$

By expanding the right-hand side of (28) up to the order two in ε one gets

$$\begin{aligned} \dot{f}^\varepsilon &= \sum_{i=1}^m \dot{\alpha}_i^\varepsilon(R) X_i(f^\varepsilon) + \frac{1}{2} \sum_{i < j} \dot{\beta}_{ij}^\varepsilon(R) [X_i, X_j](f^\varepsilon) \\ &\quad - \frac{1}{2} \left[\sum_{i=1}^m \alpha_i^\varepsilon(R) X_i, \sum_{j=1}^m \dot{\alpha}_j^\varepsilon(R) X_j \right](f^\varepsilon) + o(\varepsilon^2) \\ &= \sum_{i=1}^m \dot{\alpha}_i^\varepsilon(R) X_i(f^\varepsilon) \\ &\quad + \frac{1}{2} \sum_{i < j} \left(\dot{\beta}_{ij}^\varepsilon(R) - (\alpha_i^\varepsilon \dot{\alpha}_j^\varepsilon - \alpha_j^\varepsilon \dot{\alpha}_i^\varepsilon)(R) \right) [X_i, X_j](f^\varepsilon) \\ &\quad + o(\varepsilon^2) \end{aligned}$$

The second equality is obtained by using the anticommutativity of the Lie bracket operation. From this expression, one deduces that the property of transversality is satisfied, for ε small enough, provided that for any $R \in \mathbb{S}\mathbb{O}(m)$ the mapping h_R^ε defined by (20) is onto. This latter property was established in the proof of Theorem 1. ■

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