Feedback control of a wheeled snake mechanism with the Transverse Function approach

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***Abstract*— The Transverse Function (TF) approach is applied to the tracking control problem for a nonholonomic three-segments/snake-like wheeled mechanism similar to a planar low-dimensional version of Hirose's Active Cord Mechanism (ACM). Unlike earlier studies devoted to this type of serpentine mechanism and based on the computation and sequential application of a discrete number of open-loop control primitives, the proposed control design yields smooth (nonlinear) feedbacks in the spirit, and prolongation, of Linear Control Theory. It is also supported by a rigorous stability analysis, and it further includes a solution to the delicate –often overlooked– problem of mechanical singularities avoidance. Another asset of the approach is that the ultimate boundedness of the tracking errors, with arbitrary tracking precision obtained via the tuning of the considered transverse function parameters, is achieved for any motion of the reference frame used to specify the desired gross motion of the mechanism. These properties are illustrated by simulation results. The fact that the TF approach involves periodic functions with time-derivatives depending on frequencies used as extra control variables points out connections between this approach and biologically inspired Central Pattern Generators (CPG) often evoked in the literature on systems exhibiting internal oscillatory behavior.**

I. INTRODUCTION

We are pursuing the development of the Transverse Function (TF) approach [1] [2] for the control of highly nonlinear systems. In relation to this endeavour, the study of snake-like wheeled robots gives us the opportunity to i) apply and adapt this approach to various mechanical systems for which no feedback control solution existed so far, ii) prolong and generalize the control design methodology associated with it, and iii) propose new paradigms for the control of systems whose motion capabilities are based on the generation of oscillatory (or undulatory) shape changes.

The idea of studying biological systems via the study of man-made robotic ersatz is not new. Nor is the mirror concept of bio-observation-inspiration often invoked as an effective way to address difficult problems for which no solid theoretical corpus is yet available [3] [4]. For instance, a significant research effort, started many years ago, is devoted to the control of anthropomorphic and animal-like robots in order to better understand legged locomotion. Crawling locomotion, as examplified and perfected in Nature by snakes, is another complex locomotion mode which, despite decades of scrutiny by different scientific communities, still retains many mysteries. Of particular interest to us is the control of snake-like wheeled mechanisms, proposed by various researchers to better understand crawling locomotion (starting with the pioneering works of Hirose et al. [3] [5]). Indeed, most of the studies devoted to this theme have focused on the generation of open-loop control strategies yielding simple overall displacements along specified (and specific) directions [6], [7], [8], [9], whereas attempts to synthesize feedback control laws are few [10], incomplete and (to our point of view) mostly inconclusive due to the non-existence of adequate control design tools. One of our objectives is to show that the TF approach, and its extensions, provide such tools.

The first mechanism of this kind that we have considered is the *trident snake* system originally proposed by Ishikawa [11]. This mobile robot has a "parallel" mechanical structure and is composed of a triangular-shaped body with wheeled legs attached at its summits via rotoid articulations. The structure of the Control Lie Algebra associated with the kinematic equations of this system differs from the one of more commonly studied chained systems and gave us the idea to look for new transverse functions defined on the rotation group $\mathbb{SO}(3)$ –instead of the three-dimensional torus $\mathbb{T}^3$ [12]. The better performance observed in simulation when using these new functions comes from the fact that they better respect the system's symmetries. We have subsequently generalized the construction of such functions on $\mathbb{SO}(n)$ in relation to the case of a Control Lie Algebra maximally generated by Lie brackets of order less or equal to one [13].

The present study focuses on Hirose's *ACM III snake robot* and, more specifically, on the simplified planar model composed of three segments, as depicted on Figure 1. This system can be actuated in various ways. For instance, Ostrowski and Burdick [6] have considered the case of five control inputs: the velocities of the two articulation angles $\varphi_{1,2}$ represented on the figure, and three complementary "steering" wheel angular velocities which provide extra degrees of freedom. The more difficult case when only the articulation angular velocities can be changed is the one here addressed. It has, for instance, been considered by Ishikawa [8] to illustrate the possibility of switching between a set of piecewise sinusoidal inputs to produce a desired net displacement effect. However, to our knowledge, the tracking control problem for this type of system has not been solved previously. The facts that i) asymptotic point-stabilization is not possible by using a pure state feedback –by application of Brockett's theorem [14]–, ii) the system's Control Lie Algebra has a structure different from the one of the same dimensional chained system –and from the one of the trident snake–, iii) this system is not differentially flat so that feasible state trajectories of interest are not easily computed, and iv) mechanical singularities –which occur either when the angles $\varphi_{1,2}$ are equal or when one of them is equal to π– must not be encountered whatever the imposed gross motion of the mechanism, give an idea of the difficulties to overcome and explain in part the absence of results concerning the feedback control issue.

The paper is organized as follows. Preliminary technical recalls and notation are provided in Section II. The robot's kinematic model and error state equations are presented in Section III. The main contribution, in Section IV, specifies the control objectives and details the control design methodology based on the application of the TF approach. The validity and performance of the proposed controller are demonstrated in Section V with illustrative simulation results. Finally, the concluding Section VI points out a few research directions which could prolong the present study.

II. Notation and review

In this paper, x' denotes the transpose of a vector $x \in \mathbb{R}^n$, I_n is the identity matrix of dimension $(n \times n)$, and $O_{m\times n}$ is the zero-valued matrix of dimension $(m \times n)$. $\mathbb{T}^p$ denotes the p-dimensional torus.

A. Systems on Lie groups

Only basic properties of systems on Lie groups will be used. A few definitions and notation are recalled hereafter. The reader is referred, e.g., to [15] for more details in the context of the control of nonholonomic systems.

The tangent space of a manifold M at a point q is denoted as T_qM. If X is a vector field (v.f.) on M, the solution at time t of $\dot{x} = X(x)$ with initial condition $x(0) = q$ is denoted as $\exp(tX, q)$. A Lie group G is a manifold with a group operation $(g_1, g_2) \longmapsto g_1g_2$ such that the mapping $(g_1, g_2) \longmapsto g_1g_2^{-1}$ is smooth, with g^{-1} denoting the group inverse of g. Let G denote a connected Lie group of dimension n. The unit element of G is denoted as e, i.e. $\forall g \in G,\ ge = eg = g$. The left and right translation operators on G are denoted as L and R respectively, i.e. $\forall(\sigma, \tau) \in G^2,\ L_\sigma(\tau) = R_\tau(\sigma) = \sigma\tau$. A v.f. X on G is left-invariant iff $\forall(\sigma, \tau) \in G^2,\ dL_\sigma(\tau)X(\tau) = X(\sigma\tau)$, with df denoting the differential of a function f. The Lie algebra –of left-invariant v.f.– of the group G is denoted as $\mathfrak{g}$. If $X \in \mathfrak{g}$, $\exp(tX)$ is used as a short notation of $\exp(tX, e)$. A driftless control system $\dot{g} = \sum_{i=1}^m X_i(g)\xi_i$ on G is said to be left-invariant on G if the control v.f. X_i are left-invariant. With f, g, and h denoting smooth curves on G, one has (omitting the time index)

$$\frac{d}{dt}(gf^{-1}) = dR_{f^{-1}}(g)\left(\dot{g} - dL_{gf^{-1}}(f)\dot{f}\right) \tag{1}$$

and

$$\frac{d}{dt}(h^{-1}g) = dL_{h^{-1}}(g)\dot{g} - dR_{h^{-1}g}(e)dL_{h^{-1}}(h)\dot{h} \tag{2}$$

In the special case of the Lie group $G = \mathbb{SE}(2)$, the group operation is defined by

$$g_1g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + Q(\theta_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix} \tag{3}$$

with $g_i = (x_i, y_i, \theta_i)'$ and $Q(\theta)$ the matrix of rotation in the plane of angle θ. The unit element is $e = (0, 0, 0)'$ and the inverse of $g = (x, y, \theta)'$ is

$$g^{-1} = \begin{pmatrix} -Q(-\theta)\begin{pmatrix} x \\ y \end{pmatrix} \\ -\theta \end{pmatrix} \tag{4}$$

One deduces from (3) that

$$dL_{g_1}(g_2) = \begin{pmatrix} Q(\theta_1) & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{pmatrix} \tag{5}$$

and

$$dR_{g_2}(g_1) = \begin{pmatrix} I_2 & Q(\theta_1)\begin{pmatrix} -y_2 \\ x_2 \end{pmatrix} \\ 0_{1\times 2} & 1 \end{pmatrix} \tag{6}$$

The family $\mathbf{X} = \{X_1, X_2, X_3\}$ of v.f. defined by $X_i(g) = X(g)e_i,\ i = 1, 2, 3$ with

$$X(g) = \begin{pmatrix} Q(\theta) & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

and e_1, e_2, e_3 the canonical basis vectors of $\mathbb{R}^3$, constitutes a basis of left-invariant vector fields.

B. Transverse Functions

Notions about Transverse Functions are now recalled –see e.g. [2] for more details. Let $\mathbf{X} = \{X_1, \ldots, X_m\}$ denote a family of smooth v.f. $X_1, \ldots, X_m$ on a n-dimensional manifold M, with $m < n$. $\mathbf{X}$ satisfies the Lie Algebra Rank Condition (LARC) at some point q_0 if $\text{Lie}(\mathbf{X})(q_0) = T_{q_0}M$ with

$$\text{Lie}(\mathbf{X}) = \text{span}\{X_i, [X_i, X_j], [X_i, [X_j, X_k] \ldots \quad i, j, k, \ldots = 1, \ldots, m\}$$

and $\text{Lie}(\mathbf{X})(q) = \{X(q) : X \in \text{Lie}(\mathbf{X})\}$. Given a compact manifold K, a smooth function $f : K \longrightarrow M$ is *transverse to* $\mathbf{X}$ if, for any $\alpha \in K$,

$$\text{span}\{X_1(f(\alpha)), \ldots, X_m(f(\alpha)), df(\alpha)(T_\alpha K)\} = T_{f(\alpha)}M \tag{8}$$

with df the differential of f. Note that the dimension of K must be at least equal to $(n-m)$. Given $q_0 \in M$ such that the family $\mathbf{X}$ satisfies the LARC at q_0, the "Transverse Function theorem" in [1] ensures the existence of a family $(f^\varepsilon)_{\varepsilon>0}$ of functions transverse to $\mathbf{X}$, with $\max_\alpha \text{dist}(f^\varepsilon(\alpha), q_0) \rightarrow 0$ as $\varepsilon \rightarrow 0$, where "dist" denotes any distance locally defined in the neighborhood of q_0. Such functions can be used to achieve *practical* stabilization in a neighborhood of q_0 of the nonholonomic control system $\dot{q} = \sum_i v_iX_i(q)$, with v_i

denoting the control inputs. The idea is to stabilize q to $f^\varepsilon(\alpha)$. Thanks to (8), this is easily achieved by using $\dot\alpha$ as a (virtual) complementary control input. For instance, when $M = \mathbb{R}^n$ and $\tilde q = q - f^\varepsilon(\alpha)$ is small, one has

$$\dot{\tilde q} \approx H(\alpha)\left(\begin{smallmatrix} v \\ \dot\alpha \end{smallmatrix}\right)$$

with $H(\alpha) = (X_1(f^\varepsilon(\alpha)) \cdots X_m(f^\varepsilon(\alpha)) \;\; -\frac{\partial f^\varepsilon}{\partial\alpha}(\alpha))$ and $\left(\begin{smallmatrix} v \\ \dot\alpha \end{smallmatrix}\right) = -kH(\alpha)^\dagger \tilde q$ is a local exponential stabilizer.

III. Modeling and control objectives

The wheeled snake robot depicted on Fig. 1 is composed of three wheeled "segments" connected by two actuated rotoid articulations. From a mechanical point of view, this system is alike a unicycle vehicle pulling two trailers with off-axle trailer hitches. However, due to the existence of mechanical singularities, actuating the joint angles $\varphi_{1,2}$ is not strictly equivalent to actuating the longitudinal and angular velocities of one of the unicycles. Actually, this actuation particularity makes an important difference at the control level. It also underlies the serpentine locomotion mode which allows the system to be displaced without crossing mechanical singularities.

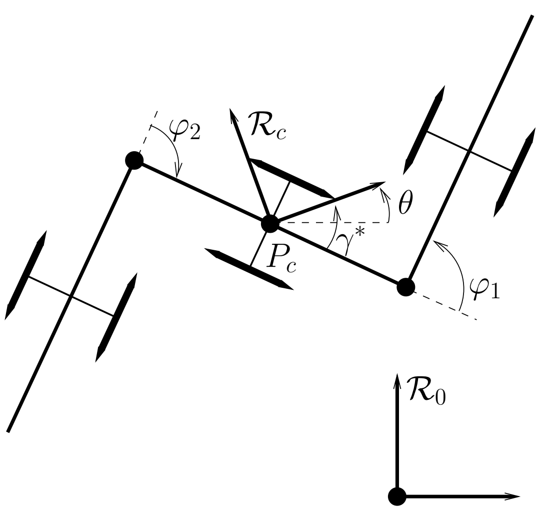

Fig. 1. Three-segments snake robot

Given an inertial frame $\mathcal{R}_0$ and a body-fixed frame $\mathcal{R}_c$, here attached for symmetry reasons to the snake's mid-segment, the configuration of this segment in Cartesian space is given by

$$g = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathbb{SE}(2)$$

with x and y the coordinates of the point P_c (the origin of $\mathcal{R}_c$ located on the wheels' axle at mid-distance of the rotoid articulations) in $\mathcal{R}_0$, and θ the orientation of $\mathcal{R}_c$ w.r.t. $\mathcal{R}_0$. The orientation angle of $\mathcal{R}_c$ w.r.t. this segment is denoted as γ^*. The choice of this angle will be discussed later on in relation to the fact that some values are better than others in terms of control and singularity avoidance.

A. Control model

The "shape" of the snake robot depends on the angle shape vector

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{T}^2$$

and the complete configuration vector of the system is thus given by $(g, \varphi) \in \mathbb{SE}(2) \times \mathbb{T}^2$.

Let v_ℓ and v_θ denote the linear and angular velocity of the mid-segment respectively, so that

$$\left\{ \begin{array}{lcl} \dot x &=& v_\ell \cos(\theta - \gamma^*) \\ \dot y &=& v_\ell \sin(\theta - \gamma^*) \\ \dot\theta &=& v_\theta \end{array} \right. \tag{9}$$

Assuming, for the sake of notation simplicity, that the distances between the three wheels'axles and adjacent rotoid articulations are all equal to one, the classical non-slipping assumption associated with wheel-ground contact yields the following two nonholonomic constraints:

$$\begin{array}{l} -v_\ell \sin\varphi_1 + v_\theta(1 + \cos\varphi_1) + \dot\varphi_1 = 0 \\ v_\ell \sin\varphi_2 - v_\theta(1 + \cos\varphi_2) + \dot\varphi_2 = 0 \end{array}$$

In matrix form these equations can be written as

$$\dot\varphi = A_\varphi(\varphi) v \tag{10}$$

with $v = (v_\ell, v_\theta)'$ and

$$A_\varphi(\varphi) = \begin{pmatrix} \sin\varphi_1 & -1 - \cos\varphi_1 \\ -\sin\varphi_2 & 1 + \cos\varphi_2 \end{pmatrix} \tag{11}$$

As long as the matrix $A_\varphi(\varphi)$ is invertible, $\dot\varphi$ and v are one-to-one related and one can view either one of these vectors as the control input. In this case Eq. (9-10) define a kinematic control system for the wheeled snake with v as the control input. Otherwise, v is no longer well defined as a function of $\dot\varphi$ and, since the *physical* control input is $\dot\varphi$, v cannot be taken as an equivalent control input at these singular configurations. One of the control objectives is to ensure that such configurations are never met. From (11)

$$\begin{array}{rcl} \det(A_\varphi(\varphi)) &=& \sin\varphi_1 - \sin\varphi_2 + \sin(\varphi_1 - \varphi_2) \\ &=& 4\sin\frac{\varphi_1 - \varphi_2}{2}\cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2} \end{array}$$

Therefore $A_\varphi(\varphi)$ is singular when either φ_1 or φ_2 is equal to π, or $\varphi_1 = \varphi_2$. The following change of coordinates

$$\varphi \longmapsto \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \tan\frac{\varphi_1}{2} \\ \tan\frac{\varphi_2}{2} \end{pmatrix} \tag{12}$$

is well defined away from $\varphi_{1,2} = \pi$ and is introduced to simplify both the avoidance of the first type of singular angles and the writing of the control model. Indeed, concerning the latter issue, one has from (10)

$$\dot\eta = A(\eta) v\,, \quad A(\eta) = \begin{pmatrix} \eta_1 & -1 \\ -\eta_2 & 1 \end{pmatrix} \tag{13}$$

so that each shape variable η_i now satisfies a linear differential equation. By regrouping relations (9) and (13) one obtains the following driftless control model (10):

$$\left\{ \begin{array}{lcl} \dot g &=& X(g) C v \\ \dot\eta &=& A(\eta) v \end{array} \right. \tag{14}$$

with $X(g)$ defined by (7) and

$$C = \begin{pmatrix} \cos\gamma^* & 0 \\ -\sin\gamma^* & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

IV. Control design

A. Basics of the control design

Given a (any) reference gross-motion for the snake body, specified by a reference frame trajectory $g_r(.) = (x_r, y_r, \theta_r)'(.)$, the control goal is to stabilize this reference trajectory (for the snake body configuration g) while avoiding singular shape values for which $A(\eta)$ is not invertible. To this purpose it is useful to also introduce a reference "shape trajectory" η_r the choice of which, beyond the fact that this trajectory must stay away from singular configurations, will be discussed later on. One can then form an "error-system" by considering the tracking errors defined by

$$\tilde{g} = g_r^{-1} g \quad , \; \tilde{\eta} = \eta - \eta_r \tag{16}$$

By using (2), (5), (6), and (14),

$$\begin{cases} \dot{\tilde{g}} &= X(\tilde{g})Cv + p_g(\tilde{g}, v_r) \\ \dot{\tilde{\eta}} &= A(\eta_r + \tilde{\eta})v - \dot{\eta}_r \end{cases} \tag{17}$$

with v_r the reference frame velocity vector defined by $\dot{g}_r = X(g_r)v_r$, and

$$p_g(\tilde{g}, v_r) = -\begin{pmatrix} 1 & 0 & -\tilde{g}_2 \\ 0 & 1 & \tilde{g}_1 \\ 0 & 0 & 1 \end{pmatrix} v_r$$

the additive "perturbation" arising from the motion of the reference frame. This error-system may also be written as

$$\dot{\xi} = Z_1(\xi)v_1 + Z_2 v_2 + p_\xi(\xi, v_r, \dot{\eta}_r) \tag{18}$$

with

$$\xi = \begin{pmatrix} \tilde{g} \\ \tilde{\eta} \end{pmatrix}, \; Z_1(\xi) = \begin{pmatrix} \cos(\xi_3 - \gamma^*) \\ \sin(\xi_3 - \gamma^*) \\ 0 \\ \xi_4 + \eta_{r,1} \\ -(\xi_5 + \eta_{r,2}) \end{pmatrix}, \; Z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \tag{19}$$

and

$$p_\xi(\xi, v_r, \eta_r) = \begin{pmatrix} p_g(\tilde{g}, v_r) \\ -\dot{\eta}_r \end{pmatrix} \tag{20}$$

Due to possible variations of η_r, the control v.f. Z_1 of this system can be time-varying. By considering the Lie brackets $Z_3 = [Z_1, Z_2]$, $Z_4 = [Z_1, Z_3]$, and $Z_5 = [Z_2, Z_3]$, one verifies that span$\{Z_i(0);\; i = 1, \ldots, 5\} = \mathbb{R}^5$ provided that $\eta_{r,1} \neq \eta_{r,2}$. Therefore, in this case, the local controllability at $\xi = 0$ of the error-system is obtained with Lie brackets of order up to two only.

The tracking control problem can now be formulated as the problem of stabilizing the origin $\xi = 0$ of the error-system. What is meant here by "stabilization" calls for a few complementary remarks and explanations. In particular, it must be noted that the *asymptotic* stabilization of the origin is not, in this case, a suitable control objective. Indeed, assuming for instance that η_r is constant, then the convergence of $\tilde{\eta}$ to zero by using a smooth feedback control would yield the convergence of v to zero, thus forbidding the stabilization of $\tilde{g} = 0$ whenever the reference velocity v_r would not itself tend to zero. Similarly, the asymptotic stabilization of $\tilde{g} = 0$ would in general imply the passage thru –or the convergence to– singular shape configurations, and thus the non-stabilization of $\tilde{\eta} = 0$. In the present case, a more appropriate –and achievable– objective is the *practical stabilization* of the origin of the error-system, i.e. the stabilization of a set within a *neighborhood* of the origin. The Transverse Function (TF) approach [1], [2] has been developped for this purpose, with the image set of an adequately chosen transverse function playing the role of the above-mentioned set. The remainder of this paper details some aspects of its application to the mechanism under consideration.

As recalled in Section II, the existence of transverse functions is guaranteed for any controllable driftless system, and explicit general expressions have been proposed in [2]. Nevertheless, the fact that different functions yield different closed-loop behaviors (some better than others) explains why the design of transverse functions is still a largely open topic. Moreover, there are also various ways to design stabilizing feedback laws based on the TF approach. For instance, a systematic method is proposed in [2] when the considered control system is invariant w.r.t. a Lie group operation. Otherwise, one can use an homogeneous (nilpotent) controllable approximation of the system (see, e.g. [16] for more details) to design both the TF and an associated feedback law. This solution is, for instance, the one reported in [12] for the control of Ishikawa's trident snake. A drawback of this solution is that it only yields, in general, local stability results. We show next that, by taking advantage of the decoupling between the dynamics on $\mathbb{SE}(2)$ and the shape dynamics, one can obtain stronger stability and convergence properties.

Let $\bar{f} : (\alpha, \eta_r) \longmapsto \bar{f}(\alpha, \eta_r)$ denote a smooth function from $K \times \mathbb{R}^2$ to $\mathbb{SE}(2) \times \mathbb{R}^2$, with K a l-dimensional compact manifold. Along any smooth curves $\alpha(.), \eta_r(.)$,

$$\dot{\bar{f}}(\alpha, \eta_r) = d_\alpha \bar{f}(\alpha, \eta_r)\dot{\alpha} + d_{\eta_r}\bar{f}(\alpha, \eta_r)\dot{\eta}_r$$

with d_α (resp. d_{η_r}) the operator of differentiation w.r.t. α (resp. η_r). The time-derivative $\dot{\alpha}$ can itself be decomposed as

$$\dot{\alpha} = Y(\alpha)\omega := \sum_{i=1}^{l} Y_i(\alpha)\omega_i \tag{21}$$

with the Y_i's denoting vector fields defined in a neighborhood of α and ω some "free" variable. Thus,

$$\dot{\bar{f}}(\alpha, \eta_r) = d_\alpha \bar{f}(\alpha, \eta_r)Y(\alpha)\omega + d_{\eta_r}\bar{f}(\alpha, \eta_r)\dot{\eta}_r \tag{22}$$

The components of $\bar{f}$ in $\mathbb{SE}(2)$ and $\mathbb{R}^2$ are denoted by $\bar{f}_g$ and $\bar{f}_\eta$ respectively, i.e. $\bar{f}(\alpha, \eta_r) = (\bar{f}_g(\alpha, \eta_r), \bar{f}_\eta(\alpha, \eta_r))$. One of the control objectives will be to make $\tilde{g}$ and $\tilde{\eta}$ converge to $\bar{f}_g$ and $\bar{f}_\eta$ respectively. To this purpose, define the error variables $z_g = \tilde{g}\bar{f}_g^{-1}$ and $z_\eta = \tilde{\eta} - \bar{f}_\eta$. From (1), (17), and (22)

$$
\begin{aligned}
\dot{z}_g &= dR_{\bar{f}_g^{-1}}(\tilde{g})\left(\dot{\tilde{g}} - dL_{z_g}(\bar{f}_g)\dot{\bar{f}}_g\right) \\
&= dR_{\bar{f}_g^{-1}}(\tilde{g})\left(X(\tilde{g})Cv - dL_{z_g}(\bar{f}_g)\dot{\bar{f}}_g + p_g(\tilde{g}, v_r)\right) \\
&= dR_{\bar{f}_g^{-1}}(\tilde{g})dL_{z_g}(\bar{f}_g)(X(\bar{f}_g)Cv - d_\alpha\bar{f}_g Y\omega - d_{\eta_r}\bar{f}_g\dot{\eta}_r) \\
&\quad + dR_{\bar{f}_g^{-1}}(\tilde{g})p_g(\tilde{g}, v_r)
\end{aligned}
$$

with the last equality obtained by using the left-invariance of the v.f. X_i's. From (16), (17), and (22)

$$\dot{z}_\eta = A(\eta_r + \tilde{\eta})v - \dot{\eta}_r - d_\alpha\bar{f}_\eta Y\omega - d_{\eta_r}\bar{f}_\eta\dot{\eta}_r$$

Regrouping the previous relations yields the system

$$\left\{ \begin{pmatrix} \dot{z}_g \\ \dot{z}_\eta \end{pmatrix} = \begin{pmatrix} B_v & B_\omega \\ A(\eta_r + \tilde{\eta}) & -d_\alpha\bar{f}_\eta Y \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix} + \begin{pmatrix} P_g \\ P_\eta \end{pmatrix} \right. \tag{23}$$

with

$$
\begin{aligned}
B_v &= dR_{\bar{f}_g^{-1}}(\tilde{g})dL_{z_g}(\bar{f}_g)X(\bar{f}_g)C \\
B_\omega &= -dR_{\bar{f}_g^{-1}}(\tilde{g})dL_{z_g}(\bar{f}_g)d_\alpha\bar{f}_g Y \\
P_g &= dR_{\bar{f}_g^{-1}}(\tilde{g})(-dL_{z_g}(\bar{f}_g)d_{\eta_r}\bar{f}_g\dot{\eta}_r + p_g) \\
P_\eta &= -\dot{\eta}_r - d_{\eta_r}\bar{f}_\eta\dot{\eta}_r
\end{aligned}
$$

The following result specifies a feedback control law for the extended control input (v, ω) which renders the origin $(z_g, z_\eta) = (0, 0)$ of this system asymptotically stable.

Theorem 1 *Assume that*

1) *For each η_r in some compact set $E \subset \mathbb{R}^2$, the function $\alpha \longmapsto \bar{f}(\alpha, \eta_r)$ is transverse to the family of vector fields $\{Z_1, Z_2\}$.*
2) *For any $(\alpha, \eta_r) \in K \times E$, $\eta_{r,1} + \bar{f}_{\eta_1}(\alpha, \eta_r) \neq \eta_{r,2} + \bar{f}_{\eta_2}(\alpha, \eta_r)$, with $\bar{f}_{\eta_1}$ and $\bar{f}_{\eta_2}$ the components of $\bar{f}_\eta$.*

and consider the feedback control law

$$\left\{ \begin{array}{l} v = A(\eta)^{-1}(d_\alpha\bar{f}_\eta Y\omega - P_\eta - k_\eta z_\eta)\,, \quad k_\eta > 0 \\ \omega = -B^\dagger(k_g z_g + P)\,, \quad k_g > 0 \end{array} \right. \tag{24}$$

with

$$
\begin{aligned}
B &= B_v A(\eta_r + \bar{f}_\eta)^{-1} d\alpha\bar{f}_\eta Y + B_\omega \\
B^\dagger &= B'(BB')^{-1}, \text{ or any right pseudo-inverse of } B \\
P &= P_g - B_v A(\eta_r + \bar{f}_\eta)^{-1}P_\eta
\end{aligned} \tag{25}
$$

Then, for any reference trajectories g_r, η_r such that the associated velocities v_r and $\dot{\eta}_r$ are bounded and $\eta_r(t) \in E\,,\ \forall t$, the origin $z = 0$ of the controlled system (23) *is asymptotically stable.*

Proof: From (23) and (24)

$$\dot{z}_\eta = -k_\eta z_\eta \tag{26}$$

so that $z_\eta = 0$ is asymptotically stable. Since $\eta = \eta_r + \tilde{\eta} = \eta_r + \bar{f}_\eta + z_\eta$, and each component of z_η exponentially decreases to zero, it follows from the theorem's second assumption that the matrix $A(\eta)^{-1}$ in (24) is well defined and bounded along any solution of the closed-loop system provided that $z_\eta(0)$ is small enough, i.e. provided that $\eta(0)$ is sufficiently close to $\eta_r(0) + \bar{f}_\eta(\alpha(0), \eta_r(0))$. From now on we assume that this condition upon $z_\eta(0)$ is satisfied.

Let us now consider the dynamics of z_g. Since $\eta = \eta_r + \bar{f}_\eta + z_\eta$, one deduces from (23) and (24) that

$$\dot{z}_g = B_v A(\eta_r + \bar{f}_\eta + z_\eta)^{-1}(d_\alpha\bar{f}_\eta Y\omega - P_\eta - k_\eta z_\eta) + B_\omega\omega + P_g$$

In view of (25), this relation may also be witten as

$$\dot{z}_g = B\omega + P + B_v R(z_\eta) \tag{27}$$

with

$$
\begin{aligned}
R(z_\eta) &= -k_\eta A(\eta_r + \bar{f}_\eta + z_\eta)^{-1}z_\eta \\
&+ \left(A(\eta_r + \bar{f}_\eta + z_\eta)^{-1} - A(\eta_r + \bar{f}_\eta)^{-1}\right)\left(d_\alpha\bar{f}_\eta Y\omega - P_\eta\right)
\end{aligned} \tag{28}
$$

Let us show that the rank of B is equal to three, so that $B^\dagger$ is well defined. From the theorem's first assumption, i.e. the property of transversality of the function $\bar{f}$, the rank of the $5 \times (2 + l)$ matrix

$$H(\alpha, \eta_r) = \begin{pmatrix} X(\bar{f}_g(\alpha, \eta_r))C & d_\alpha\bar{f}_g(\alpha, \eta_r)Y(\alpha) \\ A(\eta_r + \bar{f}_\eta(\alpha, \eta_r)) & d_\alpha\bar{f}_\eta(\alpha, \eta_r)Y(\alpha) \end{pmatrix}$$

is equal to five for any $(\alpha, \eta_r) \in K \times E$. The second assumption implies that the matrix $A(\eta_r + \bar{f}_\eta(\alpha, \eta_r))$ is invertible for any $(\alpha, \eta_r) \in K \times E$. Define

$$H_g = X(\bar{f}_g)CA(\eta_r + \bar{f}_\eta)^{-1}d_\alpha\bar{f}_\eta Y - d_\alpha\bar{f}_g Y \tag{29}$$

By pre-multiplying $H(\alpha, \eta_r)$ with the invertible matrix

$$\begin{pmatrix} I_3 & -X(\bar{f}_g(\alpha, \eta_r))CA(\eta_r + \bar{f}_\eta(\alpha, \eta_r))^{-1} \\ 0 & I_2 \end{pmatrix}$$

one deduces that

$$\text{Rank}\, H_g(\alpha, \eta_r) = 3\,, \quad \forall(\alpha, \eta_r) \in K \times E \tag{30}$$

Since $B = dR_{\bar{f}_g^{-1}}(\tilde{g})dL_{z_g}(\bar{f}_g)H_g(\alpha, \eta_r)$, it comes from (5), (6), and (30) that the rank of B is always equal to three. Therefore, the control ω in (24) is well defined and it follows from (27) that

$$\dot{z}_g = -k_g z_g + B_v R(z_\eta) \tag{31}$$

Using the fact that $\bar{f}$ takes it values in a bounded set (since this is a smooth function and (α, η_r) belongs to the compact set $K \times E$), one deduces from (4)–(7) that B_v is bounded in norm by some constant number c_0. Using the assumption that v_r and $\dot{\eta}_r$ are bounded, one then verifies from (28) that $R(z_\eta) = O_1(|z_\eta|)O_2(|z_g|) + O_3(|z_\eta|)$, with $O(|x|)$ denoting any function bounded in norm by $c|x|$ with c a constant number. Therefore

$$\dot{z}_g = -k_g z_g + c_0(O_1(|z_\eta|)O_2(|z_g|) + O_3(|z_\eta|) \tag{32}$$

and the asymptotic stability of the origin $z = 0$ follows from (26) and (32). ∎

To finalize the control design, it remains to determine a suitable transverse function, and provide some guidelines concerning the choices for γ^* and η_r.

B. Design of transverse functions

There are various ways to derive transverse functions (TFs) for a controllable driftless system. When the system has the complementary property of being left-invariant on a Lie group a general expression was proposed in [2]. The corresponding functions are defined on a torus of dimension $n-m$, with n the dimension of the state space, and m the number of control v.f. Such functions have also been considered for the control of car-like vehicles, with or without trailers, that are feedback-equivalent to chained systems [15]. In [12], [13], we proposed another design when the LARC is satisfied "at the order one", i.e. when the control v.f. and their first-order Lie brackets span the tangent space at the considered equilibrium point. It involves TFs defined on a Special Orthogonal group rather than on a torus. As explained in [13], these functions present the advantage of being endowed with symmetry properties that TFs defined on a torus do not have. However, they cannot be used directly for the error-system (17) because the satisfaction of the LARC for these systems involves second-order Lie brackets. In this section we propose new transverse functions that can be viewed as a mixt of the above-mentioned solutions in the sense that they are defined on the product of a torus and a Special Orthogonal group. Before giving their expressions for a class of 5 dimensional systems with control Lie algebras similar to the one of the snake mechanism considered here, some notation is specified.

- Δ_ε and P denote the following constant matrices:
$$\Delta_\varepsilon = \mathrm{Diag}(\varepsilon, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^3),\ P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
with $\mathrm{Diag}(x_1, \ldots, x_p)$ the diagonal matrix whose elements on the diagonal are $x_1, \ldots, x_p$.
- The following vector-valued functions are used in the calculation of the proposed transverse functions
$$\begin{aligned} \tau_{\varepsilon,\varepsilon_3}(\theta_3) &= \Delta_\varepsilon(\varepsilon_3 s\theta_3, \varepsilon_3 c\theta_3, 0, \tfrac{\varepsilon_3^3}{3} c\theta_3, -\tfrac{\varepsilon_3^3}{3} s\theta_3)' \\ \nu_{\varepsilon,\varepsilon_{54}}(Q) &= \Delta_\varepsilon(\varepsilon_{54} Q_1', \tfrac{\varepsilon_{54}^2}{2}(PQ_3)')' \end{aligned} \tag{33}$$
with $Q \in SO(3)$, Q_i the i-th column-vector of Q, $\theta_3 \in \mathbb{S}^1$, and $s\theta$ and $c\theta$ used for $\sin\theta$ and $\cos\theta$ respectively.
- Given v.f. $X_1, \ldots, X_p$ and a vector $v \in \mathbb{R}^p$, we will write Xv instead of $\sum_{i=1}^p v_i X_i$ to shorten the notation.

Theorem 2 *Let Z_1, Z_2 denote v.f. on a 5-dimensional manifold M, and q_0 a point on this manifold. Assume that the LARC is satisfied at this point with $T_{q_0}M$ spanned by the vectors $Z_1(q_0), \ldots, Z_5(q_0)$, with $Z_3 = [Z_1, Z_2]$, $Z_4 = [Z_1, Z_3]$, and $Z_5 = [Z_2, Z_3]$. Then,*

1) *There exist real numbers $\varepsilon_3, \varepsilon_{54}, \bar\varepsilon > 0$ such that, for any $\varepsilon \in (0, \bar\varepsilon)$, the function f defined on $\mathbb{S}^1 \times SO(3)$ by*
$$\begin{aligned} f(\theta_3, Q, q_0) &= f_3(\theta_3, f_{54}(Q, q_0)) \\ f_3(\theta_3, q) &= \exp(X\tau_{\varepsilon,\varepsilon_3}(\theta_3), q) \\ f_{54}(Q, q) &= \exp(X\nu_{\varepsilon,\varepsilon_{54}}(Q), q) \end{aligned} \tag{34}$$
is transverse to Z_1, Z_2.
2) *When $M = G$ is a Lie group with e its unit element, and the generating v.f. Z_1 and Z_2 are left-invariant, then the above transverse function is the group-product of elementary exponentials, i.e.*
$$f(\theta_3, Q) = f_{54}(Q, e) f_3(\theta_3, e) \tag{35}$$

The proof is omitted for lack of space. It is available from the authors upon request.

This result calls for several remarks.

1. The rationale behind the proposed TF expressions is as follows. Two functions are involved, namely f_3 and f_{54}. The role of f_3 is to grant transversality in the direction of the Lie bracket Z_3. More precisely, $\dot f_3 = Z(f_3)\mu(\theta_3)\dot\theta_3$, with the third component of the vector-valued function μ, i.e. the coefficient of $Z_3(f_3)$, always different from zero. Similarly, f_{54} grants transversality in the directions of the Lie brackets Z_4 and Z_5.

2. The proof of Theorem 2 provides useful information about the choice of ε_3 and ε_{54}. For example, one can set $\varepsilon_{54} = \varepsilon_3$ with ε_3 small enough, thus reducing the number of parameters to determine. In this case, the transversality property is satisfied for ε small enough. The suitability of the choice $\varepsilon_{54} = \varepsilon_3$, pointed out in the theorem's proof, is a consequence of the particular definition of f_3. Transversality could also be granted by considering the simpler function obtained by replacing the coefficients $\varepsilon_3^3/3$ in the expression of $\tau_{\varepsilon,\varepsilon_3}(\theta_3)$ by zeros. But the choice $\varepsilon_{54} = \varepsilon_3$ may not be suitable in this case.

3. Theorem 2 can be used, for example, to design transverse functions for the rolling sphere, which is a 5-dimensional left-invariant system on $\mathbb{R}^2 \times \mathbb{SO}(3)$. Compared to the solution proposed in [17], we have observed in simulation that this choice is preferable. This is related to the fact that it better respects the system's symmetry properties.

Let us briefly explain how Theorem 2 applies to calculate functions $\bar f$ that satisfy the assumptions of Theorem 1. Since the v.f. Z_1, Z_2 defined by (19) satisfy the assumption of Theorem 2 at $q_0 = 0$ for any constant value η_r such that $\eta_{r,1} \neq \eta_{r,2}$, relation (34) can be used to derive functions $f_{\eta_r}(\theta_3, Q, 0)$ which are transverse to $\{Z_1, Z_2\}$. The explicit calculation of these functions poses no difficulty, as the interested reader can verify by himself. It is not detailed here for lack of space. Since Z_1 depends smoothly on η_r, these functions also depend smoothly on η_r. Furthermore, f_{η_r} tends uniformly to zero as ε tends to zero. We set $\alpha = (\theta_3, Q) \in \mathbb{S}^1 \times SO(3) = K$, and $\bar f(\alpha, \eta_r) = f_{\eta_r}(\theta_3, Q, 0)$. From there, one readily verifies that $\bar f$ satisfies the assumptions of Theorem 1 for any compact set E that does not intersect the diagonal $\{\eta_r \in \mathbb{R}^2 : \eta_{r,1} = \eta_{r,2}\}$, provided that ε is small enough.

C. Choice of the reference joint angles and γ^*

As in the case of other mechanical systems moving in Cartesian space, the main control objective is to track a pre-defined reference trajectory $g_r \in \mathbb{SE}(2)$. In practice, the shape vector φ (or η) is usually less important. Nevertheless, shape singularities and collisions between body segments must be avoided. This leaves some freedom as for the choice of the reference shape trajectory η_r. In this section, we show that this choice can be related to the one of the angle γ^* (see Fig. 1) in order to limit the number and intensity of maneuvers (or shape deformations) needed to produce the desired displacements in Cartesian space. This is based on the following result.

Proposition 1 *Assume that v_r, the velocity vector associated with the reference frame trajectory g_r (i.e. $\dot{g}_r = X(g_r)v_r$), is constant and that the lateral velocity component $v_{r,2}$ is equal to zero. This corresponds to coupled longitudinal and rotation motions that a nonholonomic unicycle-type vehicle can perform. Choose*

$$\eta_r = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \tan\gamma^* \\ \dfrac{1}{\cos\gamma^*}\dfrac{\dot{\theta}_r}{v_{r,1}} \end{pmatrix} \tag{36}$$

Then, $\xi_r = (g_r', \eta_r')'$ satisfies the equality

$$\dot{\xi}_r = c_1\bar{Z}_1(\xi_r) + c_2\bar{Z}_2(\xi_r) + c_3[\bar{Z}_1, \bar{Z}_2](\xi_r) \tag{37}$$

with $c_1 = v_{r,1}\cos\gamma^*, c_2 = \dot{\theta}_r$, $c_3 = -v_{r,1}\sin\gamma^*$, and $\{\bar{Z}_1, \bar{Z}_2\}$ the control v.f. of the wheeled snake model (14).

This result, whose proof simply consists in verifying that (37) is satisfied under the assumptions of the proposition, points out a set of reference trajectories which are ”weakly nonfeasible”, in the sense that their derivatives can be decomposed along the system's control v.f. $\bar{Z}_1$ and $\bar{Z}_2$ and first-order Lie bracket $[\bar{Z}_1, \bar{Z}_2]$ only. Knowing that moving in the direction of a Lie bracket is all the more difficult that the order of the bracket is high, it matters that higher-order Lie brackets are not involved in the reference motion decomposition. Proposition 1 thus provides a simple rationale for the choice of η_r once the angle γ^* is itself determined. As for this latter choice, we are not yet aware of a simple theoretical guidance rule except that, in view of (36), this angle must be different of zero in order to obtain non-equal, and thus non-singular, reference shape values. Similarly, γ^* should not be too close to $\pi/2$, in order to keep the shape angles away from the singular value π. This still leaves us with an infinite number of possible “nominal stance” values γ^*. Simulation runs tend to indicate that “good” values, i.e. away from shape singularities and for which the transversality property is easily obtained via the choice of the transverse function parameters –recall that the transverse function depends on η_r–, are between $\pi/4$ and $\pi/3$. This latter value was used in the simulation results presented next. A thorougher study of the choice of γ^*, possibly in relation to ideas developed in [9], remains to be carried out.

V. SIMULATION RESULTS

For these simulations we have used the feedback control (24) with $k_g = 1$ and $k_\eta = 3$. The transverse function used in the control law was calculated according to (34), with vector-valued functions τ and ν depending on a few more coefficients than the functions specified in (33), in order to have some extra freedom for the “shaping” of the transverse functions. More precisely, we have used

$$\tau_{\varepsilon,\varepsilon_{31},\varepsilon_{32}}(\theta_3) = \Delta_\varepsilon(\varepsilon_{31}s\theta_3, \varepsilon_{32}c\theta_3, 0, \tfrac{\varepsilon_{31}^2\varepsilon_{32}}{3}c\theta_3, -\tfrac{\varepsilon_{31}\varepsilon_{32}^2}{3}s\theta_3)'$$
$$\nu_{\varepsilon,\varepsilon_{54},d_{2,3}}(Q) =$$
$$\Delta_\varepsilon(\varepsilon_{54}(\mathrm{Diag}(1, d_2, d_3)Q_1)', \tfrac{\varepsilon_{54}^2}{2}(P\mathrm{Diag}(d_2d_3, d_3, d_2)Q_3)')'$$

with $\varepsilon = 1$, $\varepsilon_{31} = 0.4$, $\varepsilon_{32} = 0.8$, $\varepsilon_{54} = 0.4$, $d_2 = 2$, and $d_3 = 1$. Note that smaller values could be used in order to achieve a more precise tracking. However, this would yield higher-frequency maneuvers (body deformations) and would involve larger velocity inputs. In the control calculation, we have purposely omitted to pre-compensate the terms arising from the variation of η_r in the “perturbation” vectors P_η and P (see Eq. (24)). This is to illustrate the robustness of the control law with respect to either imperfect knowledge of the reference velocities and accelerations, or deliberate simplification of the control calculation. The nominal stance angle γ^* was set equal to $\pi/3$, and the reference shape-vector η_r was calculated according to (36) with $\dot{\theta}_r/v_{r,1}$ replaced by $\dot{\theta}_r v_{r,1}/(\|v_{r,1}\|^2 + \beta)$, with β a small positive number, in order to avoid a possible division by zero. For the reported simulation, the time history of the reference frame velocity v_r is summarized in the following table.

$t \in (s)$	$v_r =$ (m/s, m/s, rad/s)$'$
$[0, 5)$	$(0, 0, 0)'$
$[5, 13)$	$(0.6, 0, 0.05(t-5))'$
$[13, 21)$	$(0.6, 0, 4 - 0.05(t-13))'$
$[21, 28)$	$(0.8, 0, 0)'$
$[28, 37)$	$(0, 0, 0.4)'$
$[37, 45)$	$(0, -0.5, 0)'$
$[45, 51)$	$(-1.5, 0, 0.6)'$
$[51, 55)$	$(-1.5, 0, -0.6)'$
$[55, 60)$	$(0, 0, 0)'$

Fig. 2 shows the evolution of the norm of z with respect to time. The peaks that can be observed on the figure correspond to –also purposely introduced– pointwise discontinuities in the time-history of v_r which, via η_r, yield discontinuities in the values of the transverse function and thus discontinuities of $z(t)$. One can also observe the subsequent exponential convergence to zero, except on the

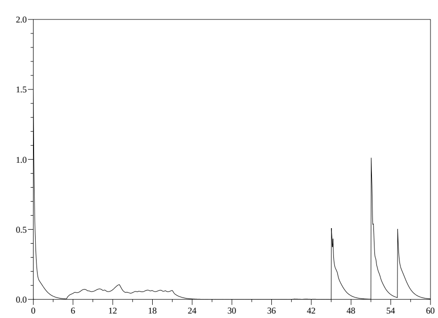

Fig. 2. $\|z\|$ vs. time

time-interval $[5, 21)$ when $\dot{\eta}_r(t) \neq 0$ is not pre-compensated. The ultimate bound of $\|z\|$ is then proportional to the size of the upperbound of the non-compensated additive perturbation and inversely proportional to the feeback gain (equal to $k_g = 1$ in the present case).

Fig. 3 shows the (x, y) trajectories of the origin of the reference frame (dotted line) and of the origin P_c of the robot's frame (dashed line). It also shows superposed snapshots, taken every ten seconds, of the wheeled mechanism and of the reference frame that it is tracking. The principle of practical tracking is well illustrated by this figure. However, only a video of the simulation can qualitatively report of the "natural" character of the mechanism's deformations in all motion phases. The time-evolution of the components of the shape angle vectors φ_r (piecewise almost constant lines) and φ (oscillatory lines) is shown on Fig. 4. The quasi-periodicity and continuous adaptation of the shape angles are noticeable. Note also that the angle amplitudes remain smaller than π and are never equal. This illutrates the fact that the mechanism's geometrical singularities are never encountered.

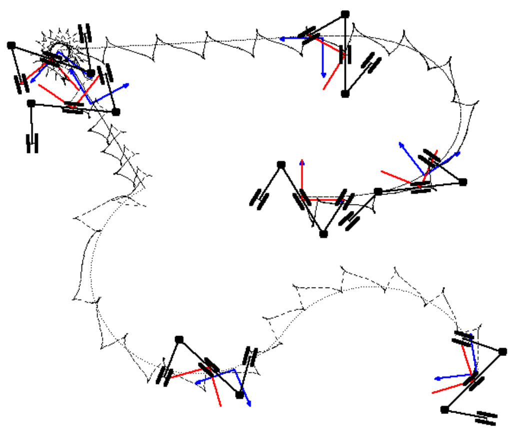

Fig. 3. Reference trajectory $(x_r(t), y_r(t))$ and snake's trajectory $(x(t), y(t))$

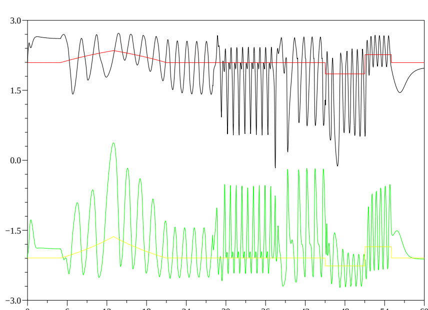

Fig. 4. Shape angles φ_r and φ vs. time

VI. Future Extensions

Ways to extend the present work are numerous. Let us just mention three issues that we plan to address:

1. Feedback control of the same mechanism with actuated steering wheels –which adds up to three control inputs. Advantages of the complementary actuation in terms of maneuverability and reduction of the control input intensities.

2. Study of several serially linked unit-mechanisms forming a snake mechanism with multi-overlapping degrees of freedom more alike Hirose's original ACM III snake robot. Consequences at the control level and for the monitoring of geometrical singularities.

3. Adaptation of the proposed control design to car-like vehicles pulling/pushing trailers with off-axle hitches –in which case $\varphi_1 = \varphi_2$ is no longer a singular configuration.

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