Observer design on the Special Euclidean group $SE(3)$

Minh-Duc Hua, Mohammad Zamani, Jochen Trumpf, Robert Mahony, Tarek Hamel

Abstract— **This paper proposes a nonlinear pose observer designed directly on the Lie group structure of the Special Euclidean group $SE(3)$. We use a gradient-based observer design approach and ensure that the derived observer innovation can be implemented from position measurements. We prove local exponential stability of the error and instability of the non-zero critical points. Simulations indicate that the observer is indeed almost globally stable as would be expected.**

I. INTRODUCTION

Estimating the pose (*i.e.*, position and attitude) of a rigid body is a key requirement for robust and high performance control of robotic vehicles. Pose estimation is a highly nonlinear problem in which the sensors normally utilized are prone to non-Gaussian noise [1]. According to a recent survey by Crassidis [2], the dominant algorithms applied to the problem of attitude estimation, Extended Kalman Filter (EKF) type methods, encounter difficulties due to the non-linearity of the state space and can display non-robustness and instability. In contrast, nonlinear observers exploit the underlying geometry in order to account for the highly nonlinear nature of the problem. As a result, they appear to be more robust and have provable almost global stability properties (see, *e.g.*, [3], [4], [5], [6], [7]). For the attitude problem, Mahony et al. [4] derived a complementary nonlinear attitude observer exploiting the underlying Lie group structure of the Special Orthogonal group $SO(3)$ of all rotations, and proved almost global stability of the error system. A locally valid symmetry-preserving nonlinear observer construction based on the Cartan moving-frame method was proposed in [8], [9]. This process is valid for arbitrary Lie groups but specializes to the same attitude filter on $SO(3)$. Lageman *et al.* [5] proposed a gradient-like observer design technique for invariant systems on Lie groups. This method leads to almost globally convergent observers given that a non-degenerate Morse-Bott cost function is used. This observer was applied to pose estimation on the Special Euclidean group $SE(3)$ from full pose measurements. Following the previous work on $SO(3)$ [4], Baldwin *et al.* [10], [11] proposed complementary observers directly on $SE(3)$ using both full state feedback and bearing only measurements of known landmarks. Vasconcelos *et al.* [7] proposed an observer that uses full range and bearing measurements of

Minh-Duc Hua and Tarek Hamel are with the laboratory I3S UNS–CRNS, Sophia Antipolis, France. E-mails: *minh.hua@polytechnique.org*, *thamel@i3s.unice.fr*.

Mohammad Zamani, Jochen Trumpf, and Robert Mahony are with the Australian National University (ANU), Australia. E-mails: *mohammad.zamani@anu.edu.au*, *jochen.trumpf@anu.edu.au*, *robert.mahony@anu.edu.au*.

known landmarks, achieving almost global asymptotic stability even when bias is present in the velocity measurements.

In this paper, we propose a nonlinear pose observer designed directly on the Lie group $SE(3)$. We use the gradient-based observer design proposed in [5] but extend this work to utilize position measurements. Following the previous work on invariant systems [9], [5] we consider left invariant kinematics along with a right invariant cost function and a right invariant Riemannian metric on $SE(3)$ and obtain autonomous error dynamics. A Lyapunov argument is used to prove local exponential stability of the proposed observer. The critical points of the error dynamics are characterized and the non-zero critical points are shown to be unstable. We go on to provide simulation studies that indicate the almost global stability of the proposed observer.

The remainder of the paper is organized as follows. Section II formally introduces the pose estimation problem on $SE(3)$ along with the notation used. Section III contains the gradient-based observer derivation and the proposed observer. Next, the stability of the observer is formally studied using Lyapunov theory in Section IV. Section V derives a discrete version of the observer to facilitate the simulation studies in Section VI. Finally, Section VII concludes the paper and some of the proofs are provided in the appendix.

II. PROBLEM FORMULATION AND NOTATION

A. Notation and mathematical identities

Let $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ denote an inertial frame attached to the earth and a body-fixed frame attached to a vehicle moving in 3D-space, respectively. The vehicle's position, expressed in $\{\mathcal{A}\}$, is denoted as $p \in \mathbb{R}^3$. The attitude of the vehicle is represented by a rotation matrix $R \in SO(3)$ of the body-fixed frame $\{\mathcal{B}\}$ relative to the inertial frame $\{\mathcal{A}\}$. Let $V \in \mathbb{R}^3$ denote the vehicle's translational velocity, expressed in $\{\mathcal{B}\}$. Let $\Omega \in \mathbb{R}^3$ denote the angular velocity, expressed in $\{\mathcal{B}\}$, of the body-fixed frame $\{\mathcal{B}\}$ with respect to the inertial frame $\{\mathcal{A}\}$.

We consider the problem of estimating the vehicle's pose which comprises the vehicle's position p and attitude R. The vehicle's pose can be interpreted as an element of the Special Euclidean group $SE(3)$, which can be represented by a matrix

$$X := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}. \tag{1}$$

This representation, commonly known as homogeneous coordinates, preserves the group structure of $SE(3)$ with the $GL(4)$ operation of matrix multiplication, *i.e.*, $X_1 X_2 \in$

$SE(3)$, for all $X_1, X_2 \in SE(3)$. The Lie-algebra $\mathfrak{se}(3)$ is the set of 4×4 matrices defined as

$$\mathfrak{se}(3) := \left\{ A \in \mathbb{R}^{4 \times 4} \mid \exists\, \Omega, V \in \mathbb{R}^3 : A = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix} \right\},$$

with $\Omega_\times$ denoting the skew-symmetric matrix associated with the cross product by Ω, *i.e.*, $\Omega_\times v = \Omega \times v$, for all $v \in \mathbb{R}^3$. The adjoint operator is a mapping $Ad : SE(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ defined as

$$Ad_X A := XAX^{-1}, \quad \text{with } X \in SE(3), A \in \mathfrak{se}(3).$$

One verifies that

$$Ad_X A = \begin{bmatrix} (R\Omega)_\times & -(R\Omega)_\times p + RV \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

For any two matrices $M_1, M_2 \in \mathbb{R}^{n \times n}$, the Euclidean matrix inner product and Frobenius norm are defined as

$$\langle\langle M_1, M_2 \rangle\rangle = \mathbf{tr}(M_1^\top M_2), \quad \|M_1\| = \sqrt{\langle\langle M_1, M_1 \rangle\rangle}.$$

Let $\mathbb{P}_a(M)$, for all $M \in \mathbb{R}^{n \times n}$, denote the anti-symmetric part of M, *i.e.*, $\mathbb{P}_a(M) = 0.5(M - M^\top)$. Let $\mathbb{P} : \mathbb{R}^{4 \times 4} \to \mathfrak{se}(3)$ denote the orthogonal projection of $\mathbb{R}^{4 \times 4}$ onto $\mathfrak{se}(3)$ with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$, *i.e.*, for all $A \in \mathfrak{se}(3)$, $M \in \mathbb{R}^{4 \times 4}$, one has

$$\langle\langle A, M \rangle\rangle = \langle\langle A, \mathbb{P}(M) \rangle\rangle = \langle\langle \mathbb{P}(M), A \rangle\rangle.$$

One verifies that for all $M_1 \in \mathbb{R}^{3 \times 3}, m_{2,3} \in \mathbb{R}^3, m_4 \in \mathbb{R}$,

$$\mathbb{P}\left(\begin{bmatrix} M_1 & m_2 \\ m_3^\top & m_4 \end{bmatrix} \right) = \begin{bmatrix} \mathbb{P}_a(M_1) & m_2 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

Let $\mathcal{M}_0$ and $\mathcal{M}_1$ denote the sub-manifolds of $\mathbb{R}^4$, respectively, defined as

$$\mathcal{M}_0 := \{ y \in \mathbb{R}^4 \mid y(4) = 0 \}, \ \mathcal{M}_1 := \{ y \in \mathbb{R}^4 \mid y(4) = 1 \}.$$

For any element $y \in \mathcal{M}_0$ or $y \in \mathcal{M}_1$, the underline notation $\underline{y} \in \mathbb{R}^3$ denotes a vector of coordinates which comprises the first three components of y, *i.e.*, $y = [\underline{y}^\top \ 0]^\top$ or $y = [\underline{y}^\top \ 1]^\top$, respectively. For later use, let us introduce some mathematical identities which can be easily verified by simple calculations.

Property 1 *For all $X \in SE(3)$, $y \in \mathcal{M}_0$, one has* $\mathbf{tr}(X^\top X y y^\top) = \mathbf{tr}(y y^\top)$.

Property 2 *For all $X \in SE(3)$, $y \in \mathcal{M}_0$, $z \in \mathcal{M}_1$, one has*

$$\mathbb{P}(X^{-\top} y z^\top) = \mathbb{P}(X y z^\top), \ \text{with } X^{-\top} := (X^{-1})^\top.$$

Property 3 *For all $y_1, y_2 \in \mathcal{M}_1$, one has* $\mathbb{P}((y_1 - y_2) y_1^\top) = \mathbb{P}((y_1 - y_2) y_2^\top)$.

B. System equations and measurements

The vehicle's pose $X \in SE(3)$, defined by Eq. (1), satisfies the differential equation

$$\dot{X} = XA, \tag{3}$$

with group velocity $A \in \mathfrak{se}(3)$. System (3) is *left invariant* in the sense that it preserves the (Lie group) invariance properties with respect to constant translation and constant rotation of the body-fixed frame $\{\mathcal{B}\}$ $X \mapsto X_0 X$.

Assume that A (*i.e.*, Ω and V) is available to measurement. Moreover, the positions of $n \in \mathbb{N}^+$ points, whose positions $\mathring{y}_i$ are constant and known in the inertial frame $\{\mathcal{A}\}$, are assumed to be measured in the body-fixed frame $\{\mathcal{B}\}$ as

$$y_i = h(X, \mathring{y}_i) := X^{-1} \mathring{y}_i, \tag{4}$$

with $y_i, \mathring{y}_i \in \mathcal{M}_1$, $i = 1, \cdots, n$. One verifies that the Lie group action $h : SE(3) \times \mathcal{M}_1 \to \mathcal{M}_1$ on the manifold $\mathcal{M}_1$ is a *right group action* in the sense that for all $X_1, X_2 \in SE(3)$ and $y \in \mathcal{M}_1$, one has $h(X_2, h(X_1, y)) = h(X_1 X_2, y)$.

III. Observer design on SE(3)

A. Gradient-based observer design

Consider an estimate $\hat{X}(t) \in SE(3)$ of the pose $X(t)$. Denote by $\hat{R}$ and $\hat{p}$ the estimates of R and p, respectively, such that $\hat{X} := \begin{bmatrix} \hat{R} & \hat{p} \\ 0 & 1 \end{bmatrix}$. Consider the observer system

$$\dot{\hat{X}} = \hat{X}(A - \alpha), \quad \hat{X}(0) \in SE(3), \tag{5}$$

with $\alpha \in \mathfrak{se}(3)$ the innovation term to be designed hereafter. Define a group error

$$E_r(\hat{X}, X) := \hat{X} X^{-1} \in SE(3). \tag{6}$$

The group error E_r is a *right invariant error* in the sense that for all $\hat{X}, X, S \in SE(3)$, one has $E_r(\hat{X}S, XS) = E_r(\hat{X}, X)$. Now, without confusion let us use the shortened notation E_r for $E_r(\hat{X}, X)$. The group error E_r provides a natural evaluation of performance of the observer response. It converges to the identity element I_4 of the group $SE(3)$ if and only if $\hat{X}$ converges to X. Using Eqs. (3) and (5), one deduces

$$\dot{E}_r = -(Ad_{\hat{X}} \alpha) E_r. \tag{7}$$

For later use, let e_i, with $i = 1, \cdots, n$, denote the estimate of $\mathring{y}_i$ which is defined as

$$e_i := h(\hat{X}^{-1}, y_i) = \hat{X} y_i, \tag{8}$$

or, equivalently,

$$e_i = h(\hat{X}^{-1}, h(X, \mathring{y}_i)) = h(X \hat{X}^{-1}, \mathring{y}_i) = E_r \mathring{y}_i. \tag{9}$$

Remark 1 *The variables e_i defined by Eq. (8), with $i = 1, \cdots, n$, can be computed by the observer.*

A recent study provides a constructive methodology of observer design for invariant systems which have the opposite invariance properties to the measurements in order to obtain well conditioned observers [5]. More precisely, Theorem 17 in [5] can be rewritten for the case of $SE(3)$ as follows.

Lemma 1 *(see [5]) Consider the **left invariant system** (3). Let $f : SE(3) \times SE(3) \to \mathbb{R}$ be a **right invariant cost function** in the sense that for all $\hat{X}, X, S \in SE(3)$, one has $f(\hat{X}S, XS) = f(\hat{X}, X)$. Let us take a **right invariant Riemannian metric** on $SE(3)$. Consider the **left observer dynamics***

$$\dot{\hat{X}} = \hat{X}A - grad_{\hat{X}} f(\hat{X}, X), \quad \hat{X}(0) \in SE(3). \tag{10}$$

Then, the dynamics of the right invariant error E_r defined by Eq. (6) is autonomous and is given by

$$\dot{E}_r = -\mathrm{grad}_{E_r} f(E_r, I_4). \qquad (11)$$

The observer system (10) is equivalent to System (5), with

$$\alpha = \hat{X}^{-1}\mathrm{grad}_{\hat{X}} f(\hat{X}, X). \qquad (12)$$

Given that we define $f(\hat{X}, X)$ such that it is minimal when $\hat{X} = X$, Lemma 1 provides a method for designing the innovation term α in order to obtain well conditioned observers. Note that since Eq. (11) is a gradient flow it is straightforward to deduce that the local minimum $E_r = I_4$ is locally asymptotically stable. In what follows, we calculate the innovation term α based on the use of the gradient decent direction of a suitable cost function.

Lemma 2 *Consider the smooth non-negative cost function $f : SE(3) \times SE(3) \to \mathbb{R}$ defined as*

$$f(\hat{X}, X) := 0.5 \sum_i k_i |(\hat{X}^{-1} - X^{-1})\mathring{y}_i|^2, \qquad (13)$$

*with k_i, $i = 1, \cdots, n$, some positive numbers. The cost function f is **right invariant** and can be expressed as a function of E_r as follows*

$$f(\hat{X}, X) = \mathcal{L}(E_r) := 0.5 \sum_i k_i |(E_r - I_4)\mathring{y}_i|^2. \qquad (14)$$

See Appendix A for the proof. For all $X \in SE(3)$, $A_1, A_2 \in \mathfrak{se}(3)$, the following equation defines a *right invariant Riemannian metric* $\langle\langle \cdot, \cdot \rangle\rangle_X$.

$$\langle\langle A_1 X, A_2 X \rangle\rangle_X := \langle\langle A_1, A_2 \rangle\rangle,$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the Euclidean metric on $\mathbb{R}^{4\times4}$. Let us calculate $\mathrm{grad}_{\hat{X}} f(\hat{X}, X)$. Using standard rules for transformations of Riemannian gradients and the fact that the Riemannian metric is right invariant, one obtains

$$\begin{aligned} \mathcal{D}_{\hat{X}} f(\hat{X}, X) \circ (\Gamma\hat{X}) &= \langle\langle \mathrm{grad}_{\hat{X}} f(\hat{X}, X), \Gamma\hat{X} \rangle\rangle_X \\ &= \langle\langle \mathrm{grad}_{\hat{X}} f(\hat{X}, X)\hat{X}^{-1}\hat{X}, \Gamma\hat{X} \rangle\rangle_X \\ &= \langle\langle \mathrm{grad}_{\hat{X}} f(\hat{X}, X)\hat{X}^{-1}, \Gamma \rangle\rangle, \end{aligned}$$

with some $\Gamma \in \mathfrak{se}(3)$. Besides, in view of Eq. (13) one has

$$\begin{aligned} \mathcal{D}_{\hat{X}} f(\hat{X}, X) \circ (\Gamma\hat{X}) &= d_{\hat{X}} f(\hat{X}, X)(\Gamma\hat{X}) \\ &= \sum_i k_i \mathring{y}_i^\top (\hat{X}^{-1} - X^{-1})^\top (-\hat{X}^{-1}(\Gamma\hat{X})\hat{X}^{-1})\mathring{y}_i \\ &= -\left\langle\left\langle \sum_i k_i \hat{X}^{-\top}(\hat{X}^{-1} - X^{-1})\mathring{y}_i\mathring{y}_i^\top, \Gamma \right\rangle\right\rangle \\ &= -\left\langle\left\langle \mathbb{P}\left(\sum_i k_i \hat{X}^{-\top}(\hat{X}^{-1} - X^{-1})\mathring{y}_i\mathring{y}_i^\top\right), \Gamma \right\rangle\right\rangle. \end{aligned}$$

Then, one deduces that

$$\begin{aligned} &\mathrm{grad}_{\hat{X}} f(\hat{X}, X) \\ &= -\mathbb{P}\left(\sum_i k_i \hat{X}^{-\top}(\hat{X}^{-1} - X^{-1})\mathring{y}_i\mathring{y}_i^\top\right)\hat{X}. \end{aligned} \qquad (15)$$

In view of Eqs. (12) and (15), the innovation term α involved in the observer system (5) satisfies

$$\alpha = -Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i \hat{X}^{-\top}(\hat{X}^{-1} - X^{-1})\mathring{y}_i\mathring{y}_i^\top\right). \qquad (16)$$

Lemma 3 *The expression (16) of α can be rewritten as*

$$\begin{aligned} \alpha &= -Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i(I_4 - E_r)\mathring{y}_i\mathring{y}_i^\top\right) \\ &= Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i(e_i - \mathring{y}_i)\mathring{y}_i^\top\right). \end{aligned} \qquad (17)$$

See Appendix B for the proof. In summary, we propose the following nonlinear observer on $SE(3)$

$$\begin{cases} \dot{\hat{X}} &= \hat{X}(A - \alpha), \quad \hat{X}(0) \in SE(3) \\ \alpha &= Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i(e_i - \mathring{y}_i)\mathring{y}_i^\top\right) \end{cases} \qquad (18)$$

with e_i given by Eq. (8) and

$$\mathbb{P}\left(\sum_i k_i(e_i - \mathring{y}_i)\mathring{y}_i^\top\right) = \begin{bmatrix} -\frac{1}{2}\sum_i k_i(\underline{e}_i \times \underline{\mathring{y}}_i)_\times & \sum_i k_i(\underline{e}_i - \underline{\mathring{y}}_i) \\ 0 & 0 \end{bmatrix}.$$

B. Group error dynamics

In order to analyze the asymptotic stability of the observer trajectory of the observer (18) to the observed system's trajectory, it is more convenient to consider the dynamics of the group error E_r and prove that its trajectory converges to the identity element of the group.

Lemma 4 *The dynamics of the group error E_r defined by Eq. (6) satisfies*

$$\dot{E}_r = \mathbb{P}\left(\sum_i k_i(I_4 - E_r)\mathring{y}_i\mathring{y}_i^\top\right) E_r. \qquad (19)$$

Furthermore, $\mathbb{P}\left(\sum_i k_i(I_4 - E_r)\mathring{y}_i\mathring{y}_i^\top\right)$ converges to zero and the equilibrium $E_r = I_4$ of System (19) is locally asymptotically stable.

See Appendix C for the proof. In the following section, we provide a more comprehensive stability analysis of the error system (19).

IV. Stability analysis

Denote $E_r = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}$, with $R_e \in SO(3)$, $p_e \in \mathbb{R}^3$. As a result of Lemma 4 and Eq. (2), one obtains

$$\begin{aligned} \dot{E}_r &= \mathbb{P}\left((I_4 - E_r)\sum_i k_i \mathring{y}_i\mathring{y}_i^\top\right) E_r \\ &= -\mathbb{P}\left(\begin{bmatrix} R_e - I_3 & p_e \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathring{\Sigma} & \mathring{\mu} \\ \mathring{\mu}^\top & \sum_i k_i \end{bmatrix}\right) E_r \\ &= -\mathbb{P}\left(\begin{bmatrix} \mathbb{P}_a(R_e\mathring{\Sigma} + p_e\mathring{\mu}^\top) & (R_e - I_3)\mathring{\mu} + \sum_i k_i p_e \\ 0 & 0 \end{bmatrix}\right) E_r \\ &= \begin{bmatrix} \Omega_{e\times} & v_e \\ 0 & 0 \end{bmatrix} E_r, \end{aligned} \qquad (20)$$

with

$$\begin{cases} \mathring{\mu} &:= \sum_i k_i \underline{\mathring{y}}_i \\ \mathring{\Sigma} &:= \sum_i k_i \underline{\mathring{y}}_i\underline{\mathring{y}}_i^\top \end{cases} \qquad (21)$$

and

$$\begin{cases} \Omega_{e\times} &:= 0.5\left(\mathring{\Sigma}R_e^\top - R_e\mathring{\Sigma} + \mathring{\mu}p_e^\top - p_e\mathring{\mu}^\top\right) \\ v_e &:= -(R_e - I_3)\mathring{\mu} - \sum_i k_i p_e \end{cases} \qquad (22)$$

System (20) is equivalent to the following system

$$\begin{cases} \dot{R}_e &= \Omega_{e\times}R_e \\ \dot{p}_e &= \Omega_{e\times}p_e + v_e \end{cases} \qquad (23)$$

which will be used hereafter for analysis purposes. Denote $E_r^* = \begin{bmatrix} R_e^* & p_e^* \\ 0 & 1 \end{bmatrix}$, with $R_e^* \in SO(3)$, $p_e^* \in \mathbb{R}^3$, as the equilibrium associated with E_r. As a consequence of Lemma 4, $\Omega_{E\times}$ and v_e defined by Eq. (22) converge to zero, which in turn implies that (using Eq. (22))

$$p_e^* = -(\textstyle\sum_i k_i)^{-1}(R_e^* - I_3)\mathring{\mu}\,, \tag{24}$$

$$\mathring{Q}R_e^{*\top} = R_e^*\mathring{Q}\,, \tag{25}$$

with $\mathring{Q}$ the symmetric matrix defined as

$$\mathring{Q} \;:=\; \mathring{\Sigma} - (\textstyle\sum_i k_i)^{-1}\mathring{\mu}\mathring{\mu}^\top\,,$$

which can be written as

$$\mathring{Q} = \left(\sum_i k_i\right)^{-1}\sum_i\sum_{j<i}k_ik_j(\underline{\mathring{y}}_i - \underline{\mathring{y}}_j)(\underline{\mathring{y}}_i - \underline{\mathring{y}}_j)^\top\,. \tag{26}$$

Since the matrices $\mathring{\Sigma}$ and $\mathring{Q}$ are symmetric, they are Hermitian and all their eigenvalues are real. Moreover, in view of Eqs. (21) and (26), it is straightforward to verify that $\mathring{\Sigma}$ and $\mathring{Q}$ are positive semi-definite. For the sake of analysis purposes, let us introduce the following assumption.

Assumption 1 *Assume that $n \geq 3$ and that the vectors $\underline{\mathring{y}}_i$, with $i = 1,\cdots,n$, are not all collinear. Assume that $rank(\mathring{\Sigma}) \geq 2$, $rank(\mathring{Q}) \geq 2$, and that the matrix $\mathring{Q}$ has three distinct eigenvalues.*

If $n \geq 3$, then it is always possible to choose a set of parameters k_i such that $rank(\mathring{\Sigma}) \geq 2$, $rank(\mathring{Q}) \geq 2$ and the three eigenvalues of $\mathring{Q}$ are distinct. From here, the main result of the present paper is stated next.

Theorem 1 *Consider System (23) and assume that Assumption 1 holds. Then,*

1) *System (23) has only four isolated equilibrium points $(R_e, p_e) = (R_{ei}^*, p_{ei}^*)$, $i = 1,\cdots,4$, with $(R_{e1}^*, p_{e1}^*) = (I_3, 0)$. For any initial condition $(R_e(0), p_e(0))$, the error trajectory $(R_e(t), p_e(t))$ converges to one of these four equilibria.*
2) *The equilibrium $(R_e, p_e) = (I_3, 0)$ is locally exponentially stable (L.E.S.).*
3) *The equilibria (R_{e2}^*, p_{e2}^*), (R_{e3}^*, p_{e3}^*), (R_{e4}^*, p_{e4}^*) are unstable.*

Proof : Let us prove Property 1 of Theorem 1. Proceeding exactly like in the proof of Theorem 5.1 in [4], one deduces from Eq. (25) and $rank(\mathring{Q}) \geq 2$ that $R_e^* = I_3$ or $\mathbf{tr}(R_e^*) = -1$. This implies that R_e^* is a symmetric matrix and, subsequently, $R_e^{*2} = I_3$. The symmetry of the matrices R_e^* and $\mathring{Q}$ yields the symmetry of the matrix $\mathring{Q}R_e^*$.

Denote the eigenvalues of $\mathring{Q}$ as λ_1, λ_2, λ_3 and assume that $0 \leq \lambda_1 < \lambda_2 < \lambda_3$. Let u_1, u_2, u_3 the associated eigenvectors of $\mathring{Q}$ such that $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \in SO(3)$. Let us denote the set $\mathbb{U}_Q \subset SO(3)$ as

$$\mathbb{U}_Q := \{R_Q \in SO(3) \mid R_Q\Lambda_Q R_Q^\top = \mathring{Q}, \Lambda_Q = \text{diag}(\lambda_i, \lambda_j, \lambda_k)\},$$

with $i, j, k \in \{1; 2; 3\}$ and distinct. This means that there are only six possibilities for Λ_Q. Then, for each possible value

of Λ_Q one verifies that there are only four possible values for $R_Q \in \mathbb{U}_Q$ as $[u_i\ u_j\ u_k]$, $[u_i\ -u_j\ -u_k]$, $[-u_i\ u_j\ -u_k]$, $[-u_i\ -u_j\ u_k]$. As a consequence, there are only 24 isolated elements in $\mathbb{U}_Q$. Then, for each value of $R_Q \in \mathbb{U}_Q$, using Eq. (25) one deduces

$$\Lambda_Q\bar{R} = \bar{R}\Lambda_Q\,, \tag{27}$$

with $\bar{R} := R_Q^\top R_e^* R_Q$ which is a symmetric matrix since R_e^* is symmetric. Eq. (27) implies that $(\lambda_i - \lambda_j)\bar{R}_{ij} = 0$, $\forall i, j \in \{1; 2; 3\}$. Since $\mathring{Q}$ has three distinct eigenvalues and two of them are not null, it follows that $\bar{R}_{ij} = 0$ for all $i \neq j$. This implies that $\bar{R}$ is diagonal. Therefore, there are only four possible values for $\bar{R}$ as

$$\bar{R}_1 := \text{diag}(1, 1, 1),\ \bar{R}_2 := \text{diag}(1, -1, -1),$$
$$\bar{R}_3 := \text{diag}(-1, 1, -1),\ \bar{R}_4 := \text{diag}(-1, -1, 1)\,.$$

Then, the matrix R_e^* can be deduced as $R_e^* = R_Q\bar{R}R_Q^\top$. As a consequence, there exist only four possible values for R_e^* which satisfy $R_e^* = R_Q\bar{R}R_Q^\top$, with $R_Q \in \mathbb{U}_Q$ and $\bar{R}$ equal to either $\bar{R}_1$, $\bar{R}_2$, $\bar{R}_3$, or $\bar{R}_4$. They are

$$\begin{cases} R_{e1}^* = I_3 \\ R_{e2}^* = u_1u_1^\top - u_2u_2^\top - u_3u_3^\top \\ R_{e3}^* = -u_1u_1^\top + u_2u_2^\top - u_3u_3^\top \\ R_{e4}^* = -u_1u_1^\top - u_2u_2^\top + u_3u_3^\top \end{cases}$$

Finally, p_{ei}^*, with $i = 1,\cdots,4$, the corresponding values for p_e^* satisfying Eq. (24), with $R_e^* = R_{ei}^*$, are uniquely defined *(end of proof of Property 1).*

Let us prove Property 2 of Theorem 1. Denoting $\bar{R}_e := R_e^\top$ and $\bar{p}_e := -R_e^\top p_e$, one deduces from Eqs. (23) and (22) that

$$\begin{cases} \dot{\bar{R}}_e = 0.5\left(\mathring{\Sigma} - \bar{R}_e\mathring{\Sigma}\bar{R}_e - \bar{p}_e\mathring{\mu}^\top + \bar{R}_e\mathring{\mu}\bar{p}_e^\top\bar{R}_e\right) \\ \dot{\bar{p}}_e = (I_3 - \bar{R}_e)\mathring{\mu} - \sum_i k_i\bar{p}_e \end{cases} \tag{28}$$

The equilibrium $(R_e, p_e) = (I_3, 0)$ of System (23) corresponds to the equilibrium $(\bar{R}_e, \bar{p}_e) = (I_3, 0)$ of System (28). Thus, it suffices to show that the equilibrium $(\bar{R}_e, \bar{p}_e) = (I_3, 0)$ of System (28) is L.E.S.. Consider a first order approximation of $(\bar{R}_e, \bar{p}_e)$ around the equilibrium $(I_3, 0)$ as $\bar{R}_e = I_3 + \tilde{r}_\times$ and $\bar{p}_e = \tilde{p}$, with $\tilde{r}, \tilde{p} \in \mathbb{R}^3$. Then, in first order approximations one obtains from Eq. (28) that

$$\begin{cases} \dot{\tilde{r}}_\times = -0.5(\tilde{r}_\times\mathring{\Sigma} + \mathring{\Sigma}\tilde{r}_\times) - 0.5(\mathring{\mu} \times \tilde{p})_\times \\ \dot{\tilde{p}} = \mathring{\mu} \times \tilde{r} - \sum_i k_i\tilde{p} \end{cases} \tag{29}$$

To prove the local *exponential* stability of the equilibrium $(\bar{R}_e, \bar{p}_e) = (I_3, 0)$ of System (28), it suffices to prove that the equilibrium $(\tilde{r}, \tilde{p}) = (0, 0)$ of the linearized system (29) is *uniformly asymptotically* stable. To this purpose, let us consider the following candidate Lyapunov function

$$\mathcal{V} := |\tilde{r}|^2 + 0.5|\tilde{p}|^2 = -0.5\mathbf{tr}(\tilde{r}_\times\tilde{r}_\times) + 0.5|\tilde{p}|^2\,.$$

Using Eq. (29), the fact that $\mathbf{tr}(u_\times v_\times) = -2u^\top v$, $\forall u, v \in \mathbb{R}^3$, and Eq. (21), one verifies that

$$\begin{aligned} \dot{\mathcal{V}} &= 0.5\mathbf{tr}(\tilde{r}_\times(\tilde{r}_\times\mathring{\Sigma} + \mathring{\Sigma}\tilde{r}_\times)) + \tilde{r}_\times(\mathring{\mu} \times \tilde{p})_\times) \\ &\quad + \tilde{p}^\top(\mathring{\mu} \times \tilde{r}) - \sum_i k_i|\tilde{p}|^2 \\ &= -\mathbf{tr}(\tilde{r}_\times\sum_i k_i\underline{\mathring{y}}_i\underline{\mathring{y}}_i^\top\tilde{r}_\times) - 2\tilde{p}^\top(\tilde{r}_\times\sum_i k_i\underline{\mathring{y}}_i) - \sum_i k_i|\tilde{p}|^2 \\ &= -\sum_i k_i(|\tilde{r} \times \underline{\mathring{y}}_i|^2 + 2\tilde{p}^\top(\tilde{r} \times \underline{\mathring{y}}_i) + |\tilde{p}|^2) \\ &= -\sum_i k_i(\tilde{r} \times \underline{\mathring{y}}_i + \tilde{p})^2 \leq 0\,. \end{aligned}$$

The resulting boundedness of $\mathcal{V}$ along any solution to the linearized system (29) yields the stability of the point $(\tilde{r}, \tilde{p}) = (0,0)$. The convergence of $\dot{\mathcal{V}}$ to zero implies that $\tilde{p}$ converges to $\mathring{\underline{y}}_i \times \tilde{r}, \forall i = 1, \cdots, n$. From here, we will show that this is possible only if $\tilde{p}$ and $\tilde{r}$ converge to zero. Let us consider two possible cases :

$\triangleright$ **Case 1:** If there exists some null vector $\mathring{\underline{y}}_i$ among the observed vectors, one deduces directly that $\tilde{p}$ converges to zero. Skipping technical arguments of minor importance, it remains to show that $\tilde{r} = 0$ is exponentially stable on the zero dynamics defined by $\tilde{p} = 0$, which is given by

$$\dot{\tilde{r}}_\times = -0.5(\tilde{r}_\times \mathring{\Sigma} + \mathring{\Sigma}\tilde{r}_\times). \tag{30}$$

Since $\mathring{\Sigma}$ is symmetric, it can be expressed as $\mathring{\Sigma} = R_\sigma \Lambda_\sigma R_\sigma^\top$, where $R_\Sigma \in SO(3)$ and $\Lambda_\sigma = \mathrm{diag}(\lambda_{\sigma 1}, \lambda_{\sigma 2}, \lambda_{\sigma 3})$, with $\lambda_{\sigma 1}, \lambda_{\sigma 2}, \lambda_{\sigma 3}$ the eigenvalues of $\mathring{\Sigma}$. Since $\mathring{\Sigma}$ is positive semi-definite and of rank greater than one, at least two eigenvalues of $\mathring{\Sigma}$ are positive. Denoting $\bar{r} := R_\sigma^\top \tilde{r}$, one verifies from (30) that $\dot{\bar{r}}_\times = -0.5(\bar{r}_\times \Lambda_\sigma + \Lambda_\sigma \bar{r}_\times) = -0.5(A_\sigma \bar{r})_\times$, or, equivalently, $\dot{\bar{r}} = -0.5 A_\sigma \bar{r}$, with

$$A_\sigma = \mathrm{diag}(\lambda_{\sigma 2} + \lambda_{\sigma 3}, \lambda_{\sigma 3} + \lambda_{\sigma 1}, \lambda_{\sigma 1} + \lambda_{\sigma 2}).$$

From here, it is straightforward to deduce the exponential stability of $\bar{r} = 0$, and subsequently, of $\tilde{r} = 0$.

$\triangleright$ **Case 2:** Let us consider the case where $\mathring{\underline{y}}_i \neq 0, \forall i = 1, \cdots, n$, and proceed the proof by contradiction. Assume that the ultimate values of $\tilde{r}$, denoted as $\tilde{r}_\infty$, is not identically null. Then, the proof proceeds as follows :

- Consider any pair of non-collinear vectors $(\mathring{\underline{y}}_i, \mathring{\underline{y}}_j)$. The fact that $\tilde{p}$ converges to $\mathring{\underline{y}}_i \times \tilde{r}$ and $\mathring{\underline{y}}_j \times \tilde{r}$ simultaneously implies that $\tilde{p}$ tends to be orthogonal to $\mathring{\underline{y}}_i$, $\mathring{\underline{y}}_j$, and $\tilde{r}$. This indicates that $\tilde{r}$ must converge to $\mathrm{span}\{\mathring{\underline{y}}_i, \mathring{\underline{y}}_j\}$ and that $\tilde{r}_\infty = \alpha_{ij}(\mathring{\underline{y}}_i - \mathring{\underline{y}}_j)$, with α_{ij} some time-varying scalar, since ultimately one has $\mathring{\underline{y}}_i \times \tilde{r}_\infty = \mathring{\underline{y}}_j \times \tilde{r}_\infty$. As a consequence, for all pairs of non-collinear vectors $(\mathring{\underline{y}}_i, \mathring{\underline{y}}_j)$, all resulting vectors $(\mathring{\underline{y}}_i - \mathring{\underline{y}}_j)$ are collinear.
- Consider any pair of collinear vectors $(\mathring{\underline{y}}_i, \mathring{\underline{y}}_j)$ and any vector $\mathring{\underline{y}}_k$ non-collinear with them. We have proven previously that $(\mathring{\underline{y}}_i - \mathring{\underline{y}}_k)$ and $(\mathring{\underline{y}}_j - \mathring{\underline{y}}_k)$ are collinear. Thus, there exist some constants $\alpha_{1,2}$ such that $\mathring{\underline{y}}_j = \alpha_1 \mathring{\underline{y}}_i$ and $(\mathring{\underline{y}}_j - \mathring{\underline{y}}_k) = \alpha_2(\mathring{\underline{y}}_i - \mathring{\underline{y}}_k)$. From here, one easily verifies that $\alpha_1 = \alpha_2 = 1$, since otherwise $\mathring{\underline{y}}_k$ is collinear with $\mathring{\underline{y}}_i$. As a consequence, for all pairs of collinear vectors $(\mathring{\underline{y}}_i, \mathring{\underline{y}}_j)$, all resulting vectors $(\mathring{\underline{y}}_i - \mathring{\underline{y}}_j)$ are null.
- From here, in view of the expression (26) of $\mathring{Q}$ and two previous items, one deduces that $\mathrm{rank}(\mathring{Q}) \leq 1$. The resulting contradiction with Assumption 1 yields $\tilde{r}_\infty = 0$ and, subsequently, $\tilde{p}_\infty = 0$ *(end of proof of Property 2)*.

Let us prove Property 3 of Theorem 1. The Lyapunov function $\mathcal{L}(E_r)$ defined in Eq. (14) can be rewritten as

$$\mathcal{L}(R_e, p_e) = 0.5 \sum_i k_i |(R_e - I_3)\mathring{\underline{y}}_i + p_e|^2$$
$$= \mathbf{tr}((I_3 - R_e)\mathring{\Sigma}) + 0.5 \sum_i k_i |p_e|^2 + p_e^\top (R_e - I_3)\mathring{\mu}. \tag{31}$$

In order to prove that (R_{e2}^*, p_{e2}^*) is unstable, let us first prove that for any neighborhood of (R_{e2}^*, p_{e2}^*), there exists some point $(R_e^\varepsilon, p_e^\varepsilon)$ in this neighborhood such that $\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) < \mathcal{L}(R_{e2}^*, p_{e2}^*)$. Now, take $(R_e^\varepsilon, p_e^\varepsilon) \in SO(3) \times \mathbb{R}^3$ of the form

$$\begin{cases} p_e^\varepsilon = p_{e2}^* + R_{e2}^* \varepsilon_p \\ R_e^\varepsilon = R_{e2}^*(I_3 + 2a_\varepsilon \varepsilon_{r\times} + 2(\varepsilon_{r\times})^2) \end{cases} \tag{32}$$

with $a_\varepsilon := \sqrt{1 - |\varepsilon_r|^2}$ and $\varepsilon_p, \varepsilon_r \in \mathbb{R}^3$ to be chosen such that their norms are positive and as small as possible. From Eqs. (31) and (32), one verifies that

$$\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) - \mathcal{L}(R_{e2}^*, p_{e2}^*) = -2\mathbf{tr}(R_{e2}^*(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\Sigma})$$
$$+ 0.5 \sum_i k_i(|\varepsilon_p|^2 + 2\varepsilon_p^\top R_{e2}^* p_{e2}^*)$$
$$+ \varepsilon_p^\top R_{e2}^*(R_{e2}^* - I_3 + 2R_{e2}^*(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2))\mathring{\mu}$$
$$+ 2 p_{e2}^{*\top} R_{e2}^*(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}.$$

Then, using the definition (24) of p_{e2}^* and the fact that $R_{e2}^{*2} = I_3$, one deduces

$$\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) - \mathcal{L}(R_{e2}^*, p_{e2}^*) = -2\mathbf{tr}(R_{e2}^*(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\Sigma})$$
$$+ 0.5 \sum_i k_i |\varepsilon_p|^2 + 2\varepsilon_p^\top(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}$$
$$- 2(\sum_i k_i)^{-1}\mathbf{tr}((R_{e2}^* - I_3)R_{e2}^*(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}\mathring{\mu}^\top)$$
$$= 0.5 \sum_i k_i |\varepsilon_p|^2 + 2\varepsilon_p^\top(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}$$
$$- 2\mathbf{tr}((a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{Q}R_{e2}^*) - 2(\sum_i k_i)^{-1}\mathbf{tr}((\varepsilon_{r\times})^2\mathring{\mu}\mathring{\mu}^\top).$$

From here, using the fact that $\mathring{Q}R_{e2}^*$ is symmetric and, subsequently, $\mathbf{tr}(\varepsilon_{r\times}QR_{e2}^*) = 0$, one verifies that

$$\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) - \mathcal{L}(R_{e2}^*, p_{e2}^*) = 2\varepsilon_p^\top(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}$$
$$+ 0.5 \sum_i k_i |\varepsilon_p|^2 - 2\mathbf{tr}((\varepsilon_{r\times})^2(\mathring{Q}R_{e2}^* + (\sum_i k_i)^{-1}\mathring{\mu}\mathring{\mu}^\top)).$$

For a given ε_r, choosing ε_p in the opposed direction of the vector of coordinates $(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}$, one obtains

$$\varepsilon_p^\top(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu} = -|\varepsilon_p| |(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}|,$$

and, subsequently,

$$\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) - \mathcal{L}(R_{e2}^*, p_{e2}^*) = -2|\varepsilon_p||(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}|$$
$$+ 0.5 \sum_i k_i |\varepsilon_p|^2 - 2\mathbf{tr}((\varepsilon_{r\times})^2(\mathring{Q}R_{e2}^* + (\sum_i k_i)^{-1}\mathring{\mu}\mathring{\mu}^\top)).$$

The objective is to prove the existence of ε_p and ε_r such that their norms can be chosen as small as possible, and that

$$0.5 \sum_i k_i |\varepsilon_p|^2 - 2|(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}| |\varepsilon_p|$$
$$- 2\mathbf{tr}((\varepsilon_{r\times})^2(\mathring{Q}R_{e2}^* + (\sum_i k_i)^{-1}\mathring{\mu}\mathring{\mu}^\top)) < 0. \tag{33}$$

Now, consider the following quadratic equation of x

$$\sum_i 0.5 k_i x^2 - 2|(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}|x$$
$$- 2\mathbf{tr}((\varepsilon_{r\times})^2(\mathring{Q}R_{e2}^* + (\sum_i k_i)^{-1}\mathring{\mu}\mathring{\mu}^\top)) = 0. \tag{34}$$

Using the relation $(\varepsilon_{r\times})^4 = -|\varepsilon_r|^2(\varepsilon_{r\times})^2$, one deduces

$$|(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2)\mathring{\mu}|^2 = \mathbf{tr}\left((-a_\varepsilon^2(\varepsilon_{r\times})^2 + (\varepsilon_{r\times})^4)\mathring{\mu}\mathring{\mu}^\top\right)$$
$$= -\mathbf{tr}\left((\varepsilon_{r\times})^2\mathring{\mu}\mathring{\mu}^\top\right).$$

Subsequently, the discriminant of Eq. (34) satisfies

$$\Delta = 4 \sum_i k_i \mathbf{tr}((\varepsilon_{r\times})^2\mathring{Q}R_{e2}^*).$$

Then, using the relations $\mathring{Q} = R_Q \Lambda_Q R_Q^\top$, $R_{e2}^* = R_Q \bar{R}_2 R_Q^\top$, and denoting $\bar{\varepsilon}_r := R_Q^\top \varepsilon_r$, one obtains

$$
\begin{aligned}
\Delta &= 4 \sum_i k_i \mathbf{tr}((\varepsilon_{r\times})^2 R_Q \Lambda_Q \bar{R}_2 R_Q^\top) \\
&= 4 \sum_i k_i \mathbf{tr}((R_Q^\top \varepsilon_r)_\times^2 \Lambda_Q \bar{R}_2) \\
&= 4 \sum_i k_i \mathbf{tr}((\bar{\varepsilon}_r)_\times^2 \operatorname{diag}(\lambda_1, -\lambda_2, -\lambda_3)) \\
&= 4 \sum_i k_i \mathbf{tr}((\bar{\varepsilon}_r \bar{\varepsilon}_r^\top - |\bar{\varepsilon}_r|^2 I_3) \operatorname{diag}(\lambda_1, -\lambda_2, -\lambda_3)) \\
&= 4 \sum_i k_i (\bar{\varepsilon}_{r1}^2(\lambda_2+\lambda_3) + \bar{\varepsilon}_{r2}^2(-\lambda_1+\lambda_3) + \bar{\varepsilon}_{r3}^2(-\lambda_1+\lambda_2)) .
\end{aligned}
$$

Choosing $\bar{\varepsilon}_{r2} = \bar{\varepsilon}_{r3} = 0$, one deduces that $|\bar{\varepsilon}_{r1}| = |\bar{\varepsilon}_r|$ and, subsequently, $\Delta = 4 \sum_i k_i (\lambda_2 + \lambda_3) |\varepsilon_r|^2 > 0$. Therefore, Eq. (34) has two distinct real solutions x_1 and x_2 (with $x_1 < x_2$, $x_2 > 0$) given by

$$
x_{1,2} = \left(\sum_i k_i \right)^{-1} \left(2|(a_\varepsilon \varepsilon_{r\times} + (\varepsilon_{r\times})^2) \mathring{\mu}| \pm \sqrt{\Delta} \right) .
$$

Then, choosing any $|\varepsilon_p|$ such that $\max(x_1, 0) < |\varepsilon_p| < x_2$ one ensures that inequality (33) is satisfied. Besides, one easily verifies that $|\varepsilon_p|$ and $|\varepsilon_r|$ can be chosen positive and as small as possible by using the fact that $\lim_{|\varepsilon_r| \to 0^+} x_{1,2} = 0$.

We have proven that for any neighborhood of (R_{e2}^*, p_{e2}^*), there exists some point $(R_e^\varepsilon, p_e^\varepsilon)$ in this neighborhood such that $\mathcal{L}(R_e^\varepsilon, p_e^\varepsilon) < \mathcal{L}(R_{e2}^*, p_{e2}^*)$. This, together with the non-increasing of $\mathcal{L}$ (as proved in Appendix C) and Property 1 of the theorem, implies that the observer trajectory $(R_e(t), p_e(t))$ starting from $(R_e^\varepsilon, p_e^\varepsilon)$ will never reach the equilibrium (R_{e2}^*, p_{e2}^*) and will quit this neighborhood to reach asymptotically one of the other three equilibria. This implies the instability of (R_{e2}^*, p_{e2}^*). The proof of instability of the equilibria (R_{e3}^*, p_{e3}^*) and (R_{e4}^*, p_{e4}^*) proceeds analogously. $\qquad\square$

V. EXTENSIONS FOR IMPLEMENTATION PURPOSES

A. Observer in quaternion form

For a more explicit form of the observer, one can verify that $\underline{e}_i = \hat{R}\underline{y}_i + \hat{p}$ and System (18) can be rewritten as

$$
\begin{cases}
\dot{\hat{R}} &= \hat{R}\Omega_\times + \hat{\omega}_\times \hat{R} \\
\dot{\hat{p}} &= \hat{R}V + \hat{\omega}_\times \hat{p} - \sum_i k_i(\hat{R}\underline{y}_i + \hat{p} - \mathring{\underline{y}}_i) \\
\hat{\omega} &:= 0.5 \sum_i k_i((\hat{R}\underline{y}_i + \hat{p}) \times \mathring{\underline{y}}_i) \\
\hat{R}(0) &\in SO(3), \ \hat{p}(0) \in \mathbb{R}^3
\end{cases} \tag{35}
$$

In practice, since it is difficult to preserve the evolution of $\hat{R}$ on $SO(3)$ due to numerical errors, the group of unit quaternions is a good alternative. Let $\hat{q}$ denote the unit quaternion associated with $\hat{R}$ such that

$$
\hat{R} = \hat{R}(\hat{q}) := I_3 + 2\hat{q}_0 \hat{\mathbf{q}}_\times + 2\hat{\mathbf{q}}_\times^2 , \tag{36}
$$

where $\hat{q} = [q_0, \hat{\mathbf{q}}]^\top$, $q_0 \in \mathbb{R}$ and $\hat{\mathbf{q}} \in \mathbb{R}^3$ are the real and pure parts of $\hat{q}$, respectively. Using standard rules for quaternion parametrizations and differentials (see, e.g., [12, Ch.1]), one deduces from the differential equation of $\hat{R}$ in Eq. (35) that

$$
\dot{\hat{q}} = 0.5(\mathrm{A}(\Omega) + \mathrm{B}(\hat{\omega}))\hat{q}, \tag{37}
$$

where the mappings $\mathrm{A}, \mathrm{B} : \mathbb{R}^3 \to \mathbb{R}^{4\times4}$ are defined as

$$
\mathrm{A}(\omega) := \begin{bmatrix} 0 & -\omega^\top \\ \omega & -\omega_\times \end{bmatrix}, \ \mathrm{B}(\omega) := \begin{bmatrix} 0 & -\omega^\top \\ \omega & \omega_\times \end{bmatrix}, \ \forall \omega \in \mathbb{R}^3 ,
$$

and the term $\hat{R}$, which is involved in the definition of $\hat{\omega}$ in Eq. (35), is calculated according to Eq. (36).

B. Numerical integration

If the sample time τ is small enough, then one can approximately assume that Ω and $\hat{\omega}$ remain constant in every period of time $[k\tau, (k+1)\tau]$, $\forall k \in \mathbb{N}$. Let us denote these values as Ω_k and $\hat{\omega}_k$, respectively. Note that

$$
\hat{\omega}_k = 0.5 \sum_i k_i \left((\hat{R}(\hat{q}_k)\underline{y}_{i_k} + \hat{p}_k) \times \mathring{\underline{y}}_{i_k} \right) .
$$

Then, by exact integration of Eq. (37), one obtains

$$
\hat{q}_{k+1} = \exp(0.5\tau(\mathrm{A}(\Omega_k) + \mathrm{B}(\hat{\omega}_k))) \hat{q}_k .
$$

By simple calculations, one verifies that $\mathrm{A}(x)\mathrm{B}(y) = \mathrm{B}(y)\mathrm{A}(x)$, $\forall x, y \in \mathbb{R}^3$, which implies that $\exp(\mathrm{A}(x)\mathrm{B}(y)) = \exp(\mathrm{A}(x))\exp(\mathrm{B}(y))$. Thus, one obtains

$$
\hat{q}_{k+1} = \exp(0.5\tau\mathrm{A}(\Omega_k)) \exp(0.5\tau\mathrm{B}(\hat{\omega}_k)) \hat{q}_k .
$$

Using the fact that, $\forall \omega \in \mathbb{R}^3$, $\mathrm{A}(\omega)^2 = \mathrm{B}(\omega)^2 = -|\omega|^2 I_4$, the Taylor series expansion yields

$$
\begin{aligned}
\exp(0.5\tau\mathrm{A}(\Omega_k)) &= \cos(\tfrac{\tau}{2}|\Omega_k|)I_4 + \tfrac{\tau}{2}\operatorname{sinc}(\tfrac{\tau}{2}|\Omega_k|)\mathrm{A}(\Omega_k), \\
\exp(0.5\tau\mathrm{B}(\hat{\omega}_k)) &= \cos(\tfrac{\tau}{2}|\hat{\omega}_k|)I_4 + \tfrac{\tau}{2}\operatorname{sinc}(\tfrac{\tau}{2}|\hat{\omega}_k|)\mathrm{B}(\hat{\omega}_k),
\end{aligned}
$$

with $\operatorname{sinc}(s) := \sin(s)/s$, $\forall s \in \mathbb{R}$. Therefore, the discrete version of Eq. (37) is given by

$$
\begin{aligned}
\hat{q}_{k+1} = &\left(\cos(\tfrac{\tau}{2}|\Omega_k|)I_4 + \tfrac{\tau}{2}\operatorname{sinc}(\tfrac{\tau}{2}|\Omega_k|)\mathrm{A}(\Omega_k) \right) \cdot \\
&\to \left(\cos(\tfrac{\tau}{2}|\hat{\omega}_k|)I_4 + \tfrac{\tau}{2}\operatorname{sinc}(\tfrac{\tau}{2}|\hat{\omega}_k|)\mathrm{B}(\hat{\omega}_k) \right) \hat{q}_k .
\end{aligned} \tag{38}
$$

Finally, to the second equation in (35), one can apply Euler's integration method to obtain the following discrete update equation for $\hat{p}$

$$
\begin{aligned}
\hat{p}_{k+1} = &\hat{p}_k \\
&+ \tau\left(\hat{R}(\hat{q}_k)V_k + \hat{\omega}_k \times \hat{p}_k - \sum_i k_i\left(\hat{R}(\hat{q}_k)\underline{y}_{i_k} + \hat{p}_k - \mathring{\underline{y}}_{i_k} \right) \right).
\end{aligned} \tag{39}
$$

In the next section, the reported simulation results are based on the discrete update equations (38)–(39).

VI. SIMULATION RESULTS

We have performed a suite of simulations using the discrete equations derived in Section V. Our simulations indicate excellent performance of the proposed observer in all the situations considered which reconfirms our local exponential convergence proof. Furthermore, the proposed observer converges asymptotically, in of all the simulation setups considered which indicates the almost global asymptotic stability of the filter. Our setups included various combinations of measurement error levels and initial values for the pose system (3) and (4). Error signals correspond to the measured angular velocity Ω, linear velocity V and output $\{\underline{y}_i\}$. Three orthogonal reference vectors $\{\mathring{\underline{y}}_i\}$ are assumed to be available in order to satisfy Assumption 1 and several initial values were considered for the attitude R and the position p.

Figure VI illustrates the tracking performance of the proposed observer in a situation which is typical of our simulations. Here, normally distributed noises of variance 0.1, 0.01 and 0.1 are imposed on the measurement $\{\underline{y}_i\}$,
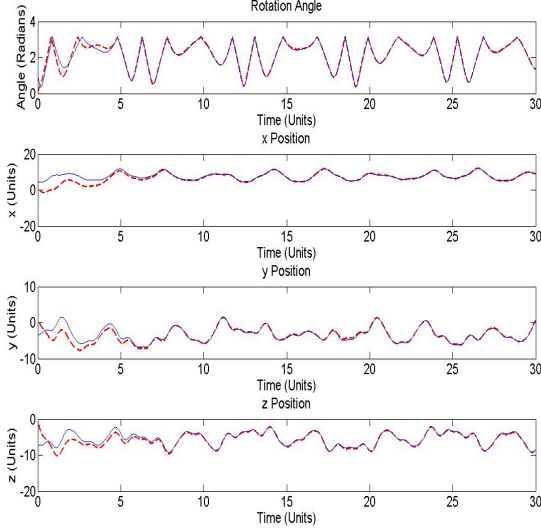
Fig. 1. The rotation angle and the position tracking performance of the proposed filter. Note that the dashed line is the estimated trajectory while the solid line represents the true trajectory.

angular velocity Ω and the linear velocity V, respectively. The proposed filter is initialized at the origin while the true trajectories are initialized differently. Note that sinusoidal inputs are considered for both the angular and the linear velocity inputs of the system. The rotation angle associated with the axis-angle representation is representing the attitude trajectory. As can be seen in Figure VI the filter trajectories converge to the true trajectory after a short transition period.

VII. Conclusions

In this paper, a nonlinear observer designed directly on the Special Euclidean group $SE(3)$ is proposed. It is a gradient-based observer that utilizes position measurements to update its state estimate. In the present work, we provide a proof for local exponential stability of the observer and instability of the undesired critical points. The proposed filter performs well in the simulations which indicates almost global asymptotic stability of the proposed observer.

VIII. Acknowledgements

References

[1] G. D. Baldwin, "Inertial vision pose estimation using non-linear observers," Ph.D. dissertation, The Australian National University, 2009.
[2] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 12–28, 2007.
[3] J. Thienel and R. M. Sanner, "A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise," *IEEE Trans. on Automatic Control*, vol. 48, no. 11, pp. 2011–2015, 2003.
[4] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the Special Orthogonal group," *IEEE Trans. on Automatic Control*, vol. 53(5), pp. 1203–1218, 2008.
[5] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-like observers for invariant dynamics on a Lie group," *IEEE Trans. on Automatic Control*, vol. 55(2), pp. 367–377, 2010.
[6] M.-D. Hua, "Attitude estimation for accelerated vehicles using GPS/INS measurements," *Control Engineering Practice*, pp. 723–732, 2010.
[7] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on SE(3) using landmark measurements," *Systems & Control Letters*, vol. 59, pp. 155–166, 2010.
[8] S. Bonnabel, P. Martin, and P. Rouchon, "Symmetry-preserving observers," *IEEE Trans. on Automatic Control*, vol. 53, no. 11, pp. 2514–2526, 2008.
[9] S. Bonnabel, P. Martin, and P. Rouchon, "Non-linear symmetry-preserving observers on Lie groups," *IEEE Trans. on Automatic Control*, vol. 54(7), pp. 1709–1713, 2009.
[10] G. Baldwin, R. Mahony, J. Trumpf, T. Hamel, and T. Cheviron, "Complementary filter design on the Special Euclidean group SE(3)," in *European Control Conference*, 2007, pp. 3763–3770.
[11] G. Baldwin, R. Mahony, and J. Trumpf, "A nonlinear observer for 6 DOF pose estimation from inertial and bearing measurements," in *IEEE Conf. on Robotics and Automation*, 2009, pp. 2237–2242.
[12] C. Samson, M. L. Borgne, and B. Espiau, *Robot Control: The Task Function Approach (Oxford Engineering Science Series No. 22)*. Oxford University Press, 1991.

Appendix

A. Proof of Lemma 2

Using Property 1 and the fact that $(\hat{X}^{-1} - X^{-1})\mathring{y}_i \in \mathcal{M}_0$, one deduces that for all $\hat{X}, X, S \in SE(3)$,

$$f(\hat{X}S, XS) = 0.5 \sum_i k_i |S^{-1}(\hat{X}^{-1} - X^{-1})\mathring{y}_i|^2$$
$$= 0.5 \sum_i k_i \mathbf{tr}(S^{-\top}S^{-1}(\hat{X}^{-1} - X^{-1})\mathring{y}_i \mathring{y}_i^\top (\hat{X}^{-1} - X^{-1})^\top)$$
$$= 0.5 \sum_i k_i \mathbf{tr}((\hat{X}^{-1} - X^{-1})\mathring{y}_i \mathring{y}_i^\top (\hat{X}^{-1} - X^{-1})^\top) = f(\hat{X}, X)$$

Using Property 1 again, one obtains

$$f(\hat{X}, X) = 0.5 \sum_i k_i \mathbf{tr}((\hat{X}^{-1} - X^{-1})\mathring{y}_i \mathring{y}_i^\top (\hat{X}^{-1} - X^{-1})^\top)$$
$$= 0.5 \sum_i k_i \mathbf{tr}(\hat{X}^\top \hat{X}(\hat{X}^{-1} - X^{-1})\mathring{y}_i \mathring{y}_i^\top (\hat{X}^{-1} - X^{-1})^\top)$$
$$= 0.5 \sum_i k_i \mathbf{tr}((I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top (I_4 - E_r)^\top)$$
$$= 0.5 \sum_i k_i |(E_r - I_4)\mathring{y}_i|^2 .$$

B. Proof of Lemma 3

Using Property 2 and the fact that $(\hat{X}^{-1} - X^{-1})\mathring{y}_i \in \mathcal{M}_0$, $\mathring{y}_i \in \mathcal{M}_1$, one verifies from (16) that

$$\alpha = -Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i \hat{X}(\hat{X}^{-1} - X^{-1})\mathring{y}_i \mathring{y}_i^\top\right)$$
$$= -Ad_{\hat{X}^{-1}}\mathbb{P}\left(\sum_i k_i (I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top\right) .$$

Finally, the second equality of (17) is deduced using (9).

C. Proof of Lemma 4

Eq. (19) can be directly deduced from Eq. (7) and Lemma 3. Then, from Eq. (11) and Property 3 one verifies that the time-derivative of the candidate Lyapunov function $\mathcal{L}(E_r)$ defined by Eq. (14) satisfies

$$\dot{\mathcal{L}}(E_r)$$
$$= \left\langle\!\left\langle \mathbb{P}\left(\sum_i k_i (I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top\right), \sum_i k_i (E_r \mathring{y}_i - \mathring{y}_i)(E_r \mathring{y}_i)^\top \right\rangle\!\right\rangle$$
$$= \left\langle\!\left\langle \mathbb{P}\left(\sum_i k_i (I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top\right), \mathbb{P}\left(\sum_i k_i (E_r \mathring{y}_i - \mathring{y}_i)(E_r \mathring{y}_i)^\top\right) \right\rangle\!\right\rangle$$
$$= \left\langle\!\left\langle \mathbb{P}\left(\sum_i k_i (I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top\right), \mathbb{P}\left(\sum_i k_i (E_r \mathring{y}_i - \mathring{y}_i)\mathring{y}_i^\top\right) \right\rangle\!\right\rangle$$
$$= -\|\mathbb{P}\left(\sum_i k_i (I_4 - E_r)\mathring{y}_i \mathring{y}_i^\top\right)\|^2 .$$

From here, the application of LaSalle's theorem allows us to conclude to proof.