Motor Control, 2012, 16, 195-228

Reconstruction of the Unknown Optimization Cost Functions from Experimental Recordings During Static Multi-Finger Prehension

Xun Niu, Alexander V. Terekhov, Mark L. Latash, and Vladimir M. Zatsiorsky

The goal of the research is to reconstruct the unknown cost (objective) function(s) presumably used by the neural controller for sharing the total force among individual fingers in multifinger prehension. The cost function was determined from experimental data by applying the recently developed Analytical Inverse Optimization (ANIO) method (Terekhov et al. 2010). The core of the ANIO method is the Theorem of Uniqueness that specifies conditions for unique (with some restrictions) estimation of the objective functions. In the experiment, subjects (n = 8) grasped an instrumented handle and maintained it at rest in the air with various external torques, loads, and target grasping forces applied to the object. The experimental data recorded from 80 trials showed a tendency to lie on a 2-dimensional hyperplane in the 4-dimensional finger-force space. Because the constraints in each trial were different, such a propensity is a manifestation of a neural mechanism (not the task mechanics). In agreement with the Lagrange principle for the inverse optimization, the plane of experimental observations was close to the plane resulting from the direct optimization. The latter plane was determined using the ANIO method. The unknown cost function was reconstructed successfully for each performer, as well as for the group data. The cost functions were found to be quadratic with nonzero linear terms. The cost functions obtained with the ANIO method yielded more accurate results than other optimization methods. The ANIO method has an evident potential for addressing the problem of optimization in motor control.

Keywords: Grasping, prehension synergy, motor redundancy, inverse optimization, Uniqueness Theorem, Principal Component Analysis

When manipulating hand-held objects, people adjust the digit forces to the weight of the handle, external torque, friction conditions, and object geometry. The task of holding an object is highly redundant: In 5-digit grasps there are 30 variables

---

Niu is with the Department of Kinesiology and the Department of Statistics, The Pennsylvania State University, University Park, PA. Terekhov is with the Department of Kinesiology, The Pennsylvania State University, University Park, PA and Institut des Systèmes Intelligents et de Robotique, CNRS-UPMC, Pyramide ISIR, Paris, France. Latash and Zatsiorsky are with Department of Kinesiology, The Pennsylvania State University, University Park, PA.

to control (3 force and 3 moment components at each contact times 5 digits) while a rigid object in space has only 6 degrees of freedom. However, when performing a prehension task performers do this in similar ways (Zatsiorsky et al. 2004). Such a similarity agrees with an idea that the central controller selects the force sharing pattern based on some unknown optimization criteria (Crowninshield and Brand 1981; Tsirakos et al. 1997; Prilutsky and Zatsiorsky 2002, Todorov 2004).

For the multifinger prehension, several candidate objective functions have been suggested and tested in the literature (Zatsiorsky et al. 2002b; Pataky et al. 2004b; Aoki et al. 2006; Niu et al. 2009). While some of them worked better than others, i.e., their application resulted in a better correspondence with the experimental data, all of them were based on guesses made by the researchers. In other words, the inverse optimization problem—finding a cost function from experimental data—was not considered and replaced by comparing the functioning of various optimization criteria in direct optimization. Such a try-and-compare approach has been common in biomechanics of human motion (for reviews see Collins 1995[AUQ1]; Tsirakos et al. 1997; Prilutsky 2000; Engelbrecht 2001; Todorov 2004, Erdemir et al. 2007[AUQ2]) since analytical inverse optimization methods did not exist. Another approach consists in selecting, a priori, a parameterized class of putative objective functions, e.g., an assumption is made that the function is linear or quadratic, and then the function parameters are estimated from experimental data. Such a methodology was mainly used outside the motor control area, in particular using methods of linear programming (Ng, Russell 2000; Abbeel, Ng 2004; Syed et al. 2008; Ziebart et al. 2008). An interesting new approach has been recently suggested by Körding and Wolpert (2006). We discuss this method in more detail in the Discussion.

Force distribution among digits in multifinger prehension is an example of a more general problem of distributing activity among several effectors acting in parallel (the most popular case of such a problem is sharing activity among a group of muscles serving a joint—Challis, Kerwin 1993; Herzog and Binding, 1993; Prilutsky and Zatsiorsky 2002; Ait-Haddou et al. 2004). Several attempts have been made recently to solve the inverse problem of optimization for such tasks analytically (Sieminski, 2006[AUQ3]; Bottasso et al. 2006) but these methods were not applied to multifinger grasping. As compared with other distribution problems, for instance with the muscle sharing problem, multifinger prehension provides three important research advantages: (a) all the variables of interest, e.g., digit forces and moments, moment arms, the task parameters, etc., can be directly measured; (b) the task parameters such as the object weight and geometry, applied external torques, etc. can be varied by a researcher according to the method requirements; and (c) the performance of different optimization functions can be validated experimentally.

The present research was performed with the following goals:

(1) To test applicability of the ANIO method to studying the multifinger prehension.
(2) To compare the cost functions obtained in different subjects. Do all the subjects use functions of the same class, e.g., quadratic function with linear terms?
(3) To compare the performance of the cost functions determined analytically using the ANIO method to performance of other cost functions.

The paper has the following structure. In Section 1, a recently developed analytical inverse optimization (ANIO) method (Terekhov et al. 2010; Park et al.

2010; Terekhov and Zatsiorsky 2011) is described. The ANIO is a general mathematical method. To apply the method to a particular task, the method should be adjusted. In Section 2 the experiment is described and the main experimental results are presented. Section 3 formulates the inverse optimization problem for the task of multifinger prehension and addresses the computational procedures of the ANIO method to determine a group of unknown objective functions. In Section 4 the performance of the objective function computed from the ANIO method is presented and compared with the performance of other cost functions. Section 5 is devoted to a general discussion.

Analytical Inverse Optimization (ANIO) Method

The mathematics behind the method, including the proofs of all the lemmas and theorems, are described in Terekhov et al. (2010). The Appendix to this paper presents the main Uniqueness Theorem on which the ANIO method is based. The main concepts of the ANIO are explained here in a basic form.

The *objective function* is assumed to be an unknown additive function J which is composed of unknown scalar differentiable functions g_i. An optimization problem with additive objective function is defined as:

Let J: $R^n \to R^1$

$$\text{Min: } J(x) = g_1(x_1) + g_2(x_2) + \text{- - -} + g_n(x_n) \tag{1}$$

Subject to: $CX=B$

where $X=(x_1, x_2, \ldots, x_n) \in R^n$, g_i is an unknown scalar differentiable function with $g'(\cdot)>0$ in the feasible region, C is a $k\times n$ matrix, rank $C=k$, and B is a k-dimensional vector, $k<n$. Such a problem can be referred to as <J, C>.

Lagrange Principle for the Inverse Optimization Problem

The necessary conditions for solving the direct optimization problem are known as the Lagrange principle. Since every experimentally observed point is assumed to be a result of minimization of some objective function, the Lagrange principle must hold. For the inverse optimization problem with additive objective functions and linear constraints the following formulation of the Lagrange principle is valid.

If the objective functions g_i in equation (1) are continuously differentiable then they satisfy the equation: $\breve{C}g'(X=0)$, for every experimentally observed X^*. Here $\breve{C} = I - C^T\left(CC^T\right)^{-1}C$. C^T denotes the transpose of matrix C.

Essentially Similar Objective Functions

Estimating the objective function from observations does not necessary lead to a unique solution. For example, multiplying an objective function by a positive number or adding a number does not change the optimal solution. Two objective functions are called essentially similar on the same set if for every possible constraint their optimization leads to the same result.

Problem Splitability

If minimization of the objective function can be performed independently for certain subsets of variables the problem is *splittable*. Splittabilty limits the inverse optimization. It was proven in (Terekhov et al. 2010) that the optimization problem (1) is splittable if and only if breve matrix $\breve{C}$ can be made block diagonal by reordering the rows and columns with the same indices.

The theorem of uniqueness is presented in Appendix. The theorem provides sufficient conditions for uniqueness, up to linear terms, of solutions of the inverse optimization problem. According to the theorem, if the experimentally observed values X^* form a k-dimensional hypersurface, where k is the number of constraints in the problem, then the solution of the inverse problem-is unique up to linear terms. In this case the equation provided by the Lagrange principle can be used to determine the objective function. In particular, if the experimental hypersurface can be defined by the equation $\breve{C}f'(X)=0$, where $f'(X)=\left(f_1'(x_1),\dots,f_n'(x_n)\right)^T$ and f_i are arbitrary scalar functions, then the sought functions g_i coincide with f_i up to linear terms (specific details are explained later in the text).

Experiment: Multifinger Prehension

In the experiment subjects maintained at rest a vertically oriented handle. The subjects used a prismatic grasp in which the fingers and the thumb contact the object in the same plane (the grasp plane, Figure 1).

Methods

Subjects. Eight young male subjects participated in the experiment (age 27.6 ± 3.0 yr, mass 74.7 ± 9.0 kg, height 176.3 ± 9.2 cm, hand length from the middle finger tip to the distal crease of the wrist with hand extended 18.4 ± 0.9 cm, hand width at the MCP level with hand extended 8.9 ± 0.7 cm). They were all right-hand dominant according to their hand usage during eating and writing. None of the subjects had a history of neuropathy or trauma to their upper limbs or professional training that might affect their hand function, such as playing musical instruments. All subjects gave informed consent according to the policies of the Office for Research Protections of The Pennsylvania State University.

Apparatus. Five six-component force/moment transducers (Nano-17, ATI Industrial Automation, Garner, NC, USA) were mounted on an aluminum handle at the bottom of which a horizontal aluminum bar was attached (0.7 m long), see Figure 1. The subjects were instructed to keep the handle vertical and static by watching an air bubble level placed at the top of the handle. The level (diameter: 32 mm) included a central circle (diameter: 15 mm) and an air bubble (diameter: 5mm) in the enclosed liquid. If the bubble was within the central circle, the trial was accepted by the researcher; otherwise the trial was repeated. We found that when the moment of force exerted by the subject was within ±40 Nmm (0.04 Nm) of the target moment of force, the bubble was at the edge of the central circle. Four transducers were used to measure forces and moments of force applied by the fingers, and the fifth transducer measured the force and moment of force produced by the thumb. The surface of each sensor was covered with sandpaper

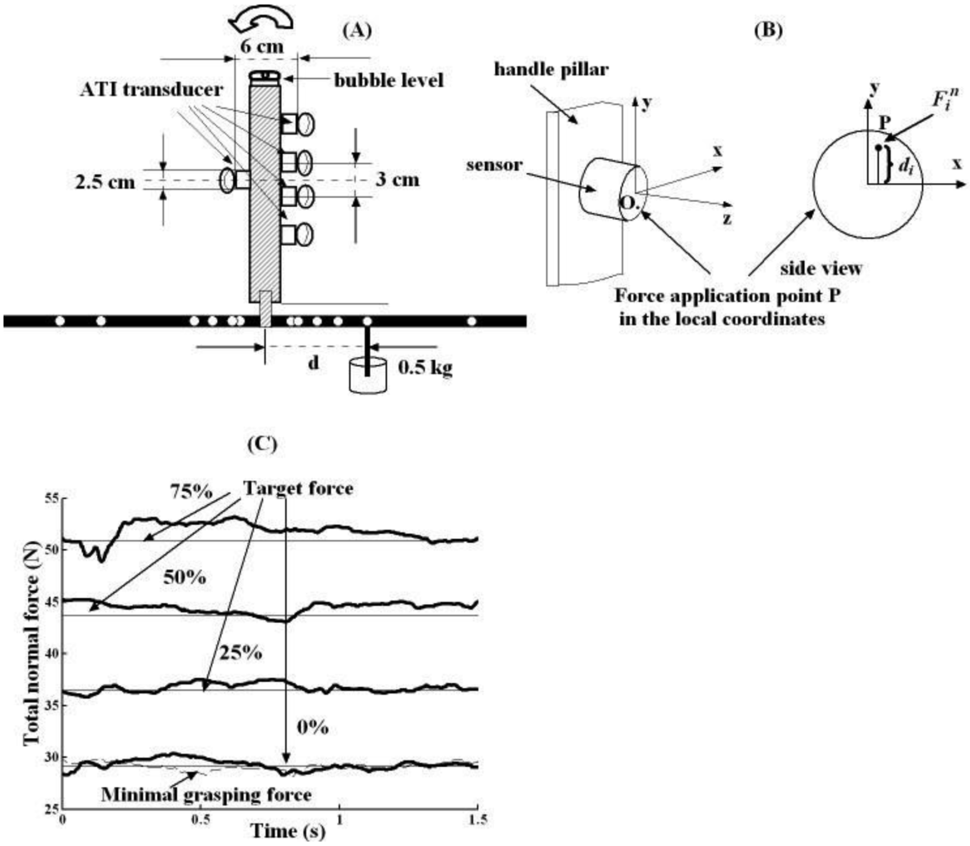


Figure 1 — An instrumented handle and recorded forces in the experiment. (A) Schematic drawing of the apparatus with five sensors and an air bubble level mounted on a handle with a T-shaped attachment. (B) Local coordinates on each transducer. (C) Total normal force generated in the tasks (examples: external torque 0.4 Nm with load of 0.5 Kg).

(the coefficient of friction =1.34 ± 0.05, Aoki et al. 2006). Sensor signals were set to zero before each trial

The distance between the centers of two finger sensors was 3.0 cm, and the thumb sensor was positioned across the midpoint between the centers of the middle and ring finger sensors. The combined mass of the handle, sensors, and the bar was 1.01 kg. Four loads, 0.25 kg, 0.5 kg, 0.75 kg and 1.0 kg could be attached at different points along the bar. Their suspension at different locations generated five external torques: 0.2 Nm and 0.4 Nm clockwise and counterclockwise, as well as a zero torque. There were a total of 20 different load/torque conditions in the experiment.

Experimental Procedure. The subjects were instructed to perform five trials at each torque/load combination. In the first trial, the subjects applied the natural normal force (grasping force), while holding the handle in the air. In the next four trials, a target line was shown on the monitor with different grasping force magnitude, which was determined by increasing the recorded natural normal thumb force by 0% (i.e., essentially repeating the first trial but with the grasping force prescribed), 25%, 50% and 75% separately in each trial. The thumb normal force was prescribed due to the necessity to apply the method to a large number of trials (see Terekhov et al. 2010).

The subjects were asked to match the target line while keeping the handle vertical without any angular or linear movement. The sequence of the four trials with the prescribed grasping force (0%, 25%, 50% and 75% of force increase) was randomized. Before the experiment, subjects were given an orientation session to familiarize themselves with the experimental tasks and apparatus. Then, subjects washed their hands to normalize skin condition.

The subject sat in a chair with the right upper arm positioned at approximately 45° abduction in the frontal plane and 30° flexion in the sagittal plane. The elbow joint was flexed approximately 90°. The forearm was pronated 90° such that the hand was in a natural grasping position. A computer monitor located in front of the subject showed the thumb normal force exerted by the subject on the handle (in statics the magnitude of the thumb normal force equals the sum of the magnitudes of the four finger forces). The subjects were coached to keep the handle vertically by looking quickly at the bubble level located at the top of the handle while mainly watching the monitor.

Subjects were instructed to take the handle from the rack and keep the handle vertically and statically in the air by monitoring the air bubble level. When the handle was stabilized and the natural grasping force was applied by the subject, data recording would start.

The subjects were told by the investigator to "Keep the handle vertical and static as precisely as you can, and match the target line by increasing or decreasing the gripping force if necessary". After the data collection in a trial stopped, the subject placed the handle back on the rack and took a 30-s break. After the subject completed all the five trials at a given torque/load combination, the investigator would change the load and/or the location of the load along the bar, and informed the subject that he could start the next trial. Sensor signals were set to zero before each trial.

The order of external torques and loads, and the force-specification percentages within each torque/load condition was randomized. Each trial took 10 s. During the experiment each subject performed 20 trials of natural grasping force exertion (4 loads×5 torque) and 80 trials of prescribed grasping force magnitude (4 loads×5 torque×4 grasping force magnitudes). The total duration of each experiment was approximately two hours.

The percentages of the actual sum of finger normal force increase were calculated across all trials, which were 3% ± 5%, 27%± 4%, 50%± 5% 74% ± 6% (mean ± *SD*) in the tasks of 0%, 25%, 50% and 75% force increase. The matching error was small.

Data Analysis. Software written in LabVIEW (National Instruments, NC, USA) was used to convert digital signals into the force and moment of force values. Data processing was performed using Matlab software package (Mathworks, In., Natick, MA, USA). The raw force/moment data were filtered with a third-order, zero-lag Butterworth low-pass filter at 10 Hz.

Since the thumb and virtual finger (VF, an imagined finger with the same mechanical effect as the combined action of all four fingers of the hand) normal forces compose a force couple and hence generate a free moment (Zatsiorsky, 2002a), the pivot point used to calculate the moment of normal force (M^n) was arbitrarily defined at the center of force application (CFA) of the thumb. Therefore, the moment arms of index, middle, ring and little fingers in producing the M^n were $y_{in} = 45 \pm d_{in} - \Delta_{th}$ mm, $y_{mi} = 15 \pm d_{mi} - \Delta_{th}$ mm, $y_{ri} = -15 \pm d_{ri} + \Delta_{th}$ mm, and $y_{li} = -45 \pm d_{li} + \Delta_{th}$ mm, respectively, where d_i is the CFA of each finger with respect to the sensor center where the finger was placed on (see Figure 1); Δ_{th} is the CFA of the thumb with respect to the thumb sensor center. The minus sign for ring and little fingers is because they generated clockwise (negative) moment of force. The method of determining d_i is described elsewhere (Niu et al. 2009).

Statistical Analysis. The Linear Mixed Model (LMM) in SPSS 16.0 (SPSS Inc., Chicago, IL, USA) was used to do the statistical tests. The LLM is more powerful to analyze repeated measures observations than other models, such as Generalized Linear Model (GLM) (Littel et al. 2006). Repeated measures (RM) ANOVA on finger normal forces were performed with the factors: TORQUE, LOAD, and TARGET FORCE. In addition, four-way RM-ANOVA was employed to analyze the effects of TORQUE, LOAD, TARGET FORCE and METHOD on the RMS differences between optimal solutions and experimental observations, where METHOD included five levels corresponding to the procedures of ANIO, MBIO, and three direct optimization methods explained in Section 4. Akaike's Information Criteria (AIC) as a criterion of goodness-of-fit was used to determine the best covariance structure for the RM-ANOVA model (Akaike 1974). We found that First Order Autoregressive Model [AR(1)] covariance structure was more appropriate to analyze the data than other covariance structures. AR(1) model assumes that the expected within-subject correlation decreases exponentially with the spacing of the factor levels between measurements. It has been also shown (Howell 2002) that AR(1) is better than Compound Symmetry (CS) covariance structure in repeated measures studies, thus the sphericity assumption testing was not required in this study. Pairwise comparison using the Bonferroni correction with family confidence coefficient .95 was calculated for the significant effects in the RM-ANOVA post hoc tests.

Experimental Results

Only results which were used in the subsequent optimization procedure—i.e., normal forces exerted by individual fingers— are reported here.

The increase of the grasping force was accomplished by the increases of all four finger normal forces. Figure 2 shows, as example, the index and little finger normal forces at various external torques, loads, and target grasping forces. As expected, the individual finger forces increased with an increase of the supported load and target grasping force. With a change of the external torque, the index and little finger forces changed in opposite directions.

Three-way RM-ANOVA showed that TORQUE, LOAD and TARGET FORCE all had significant main effects on F_{in}^n ($p < .001$ for all). Bonferroni pairwise comparison showed that increased significantly for every level of the factor TARGET FORCE from 0% to 75% ($p < .001$ for each). None of the two- and three-way interactions was significant ($p > .5$).

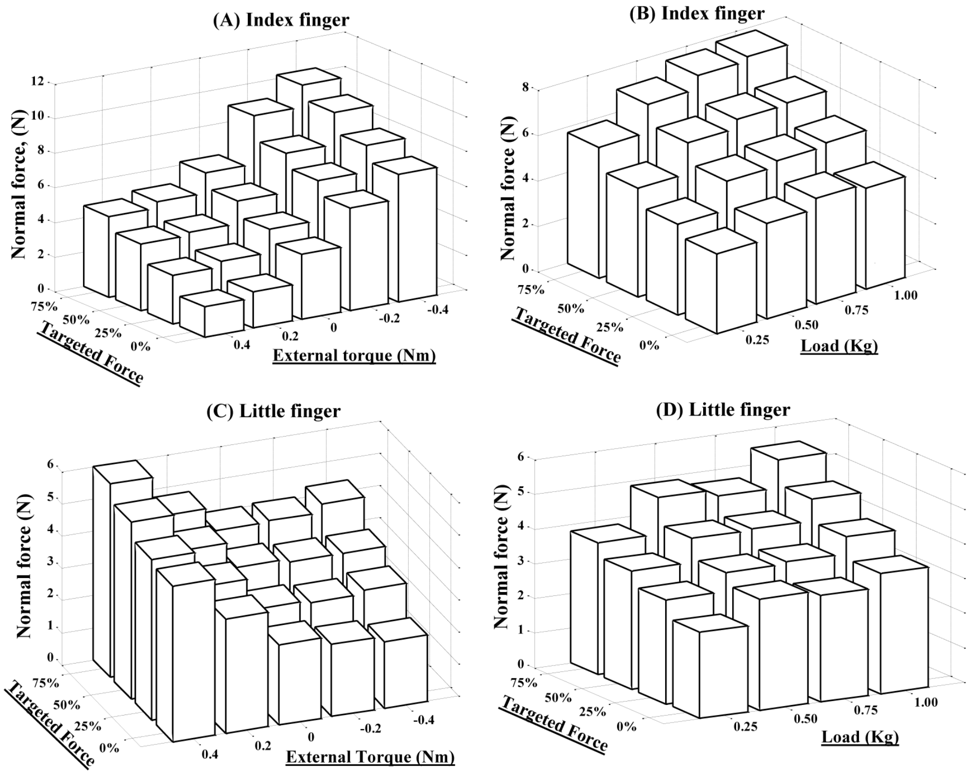


Figure 2 — Index and little finger normal forces at various torques, loads, and target levels of force. Group averages over eight subjects are shown.

Similar to the index finger force, F^n_{mi}, F^n_{ri} and F^n_{li} increased with the increase of LOAD ($p < .001$ for each), and TARGET FORCE whose effect, however, depended on the external torque. For those fingers, the interaction TORQUE×TARGET FORCE, which reflected the conjoint effect of TORQUE and TARGET FORCE on the normal forces, was statistically significant ($p < .001$ for each). No significant TORQUE×LOAD interaction effects were found for any finger. Pairwise comparisons with Bonferroni corrections showed that F^n_{mi}, F^n_{ri} and F^n_{li} increased significantly with an increase of TARGET FORCE at each level of TORQUE ($p < .05$ for each).

The moment arms of fingers with respect to the thumb, y_i (i=in, mi, ri, li), were 46.5 ± 4.3 mm, 14.4 ± 1.5 mm, –15.3 ± 2.0 mm and –45.9 ± 2.2 mm (average± *SE* across all tasks and subjects). Three-way RM-ANOVA was performed to inspect the effect of TORQUE, LOAD and TARGET FORCE on y_i. It was found y_{in} was affected significantly by TARGET FORCE ($p = .042$), with a significant difference between 75% and 0% normal force increase ($p = .029$); y_{ri} was affected by TORQUE ($p = .018$), with a significant difference between L2 and R2 ($p = .006$); y_{mi} and y_{li} were not affected by any factors significantly ($p > .051$).

The principal component analysis (PCA) was performed on the observations (80 points for the individual subjects and 640 points for all the subjects combined) . In the analysis of individual subject's data, it was found that 93.53 ± 0.43% of the total variance across subjects was accounted for by the first two principal components which had the larger eigenvalues. The first two principal components explained 88.14% variance in the pooled data across all subjects.

Therefore, it has been concluded that the experimental observations tend to lie on a 2-dimensional hyperplane in the 4-dimensional space (although the constraints, i.e., the load, the required total force and the resisted moment, were different in all 80 trials; see section 2 for more details). An example of the projections of the hyperplane onto a three-dimensional space ($F_{in}^n, F_{mi}^n, F_{li}^n$) and the finger force distribution in the space is presented in Figure 3B.

Computational Procedures of ANIO Method

The ANIO method is a general mathematical method that can be applied to many real life problems. This section explains the formulation of optimization problem for the experiment described in Section 3.3, and specifies the ANIO computational procedures for the task of static multifinger prehension.

Formulation of Optimization Problem

The conditions of static equilibrium for the studied task (described in Section 2 above) should satisfy three equality constraints (balancing external forces in the horizontal and vertical directions and balancing an external moment applied to the handle) and two inequality constraints (the normal force should be sufficiently large to prevent the object slipping from the hand and the finger forces are nonnegative and cannot be larger than the maximal finger force), for a more detailed explanation see Zatsiorsky, Latash (2009). Because the surfaces of the sensors were covered with sandpaper the friction at the finger tips was large and slipping was not a problem in the current study.

It was previously shown that sharing percentage of the total normal force among the fingers is quite reproducible across trials with fixed load force and external torque (Zatsiorsky et al. 2002 a, b; Shim et al. 2003). This observation allows assuming that particular force sharing patterns result from minimization of a certain objective function. The inverse optimization problem was defined as

$$\text{Min } J = \sum_{i=1}^{4} g_i\left(F_i^n\right)$$

$$\text{Subject to} \begin{cases} F_{in}^n + F_{mi}^n + F_{ri}^n + F_{li}^n = F_{total}^n \\ F_{in}^n Y_{in} + F_{mi}^n Y_{mi} + F_{ri}^n Y_{ri} + F_{li}^n Y_{li} = M^n \end{cases} \tag{2}$$

where $F = \left[F_{in}^n, F_{mi}^n, F_{ri}^n, F_{li}^n\right]^T$ is a 4 × 1 nonnegative vector of finger forces (the subscripts *in*, *mi*, *ri*, and *li* refer to the index, middle, ring, and little finger, respectively), superscript *n* indicates that normal forces are considered, F_{total}^n is the sum of four finger normal forces, $Y = \left[Y_{in}, Y_{mi}, Y_{ri}, Y_{li}\right]^T$ designate the moment arms of the normal finger forces, M^n is moment of normal force. The constraints can be written as *CF=B*.

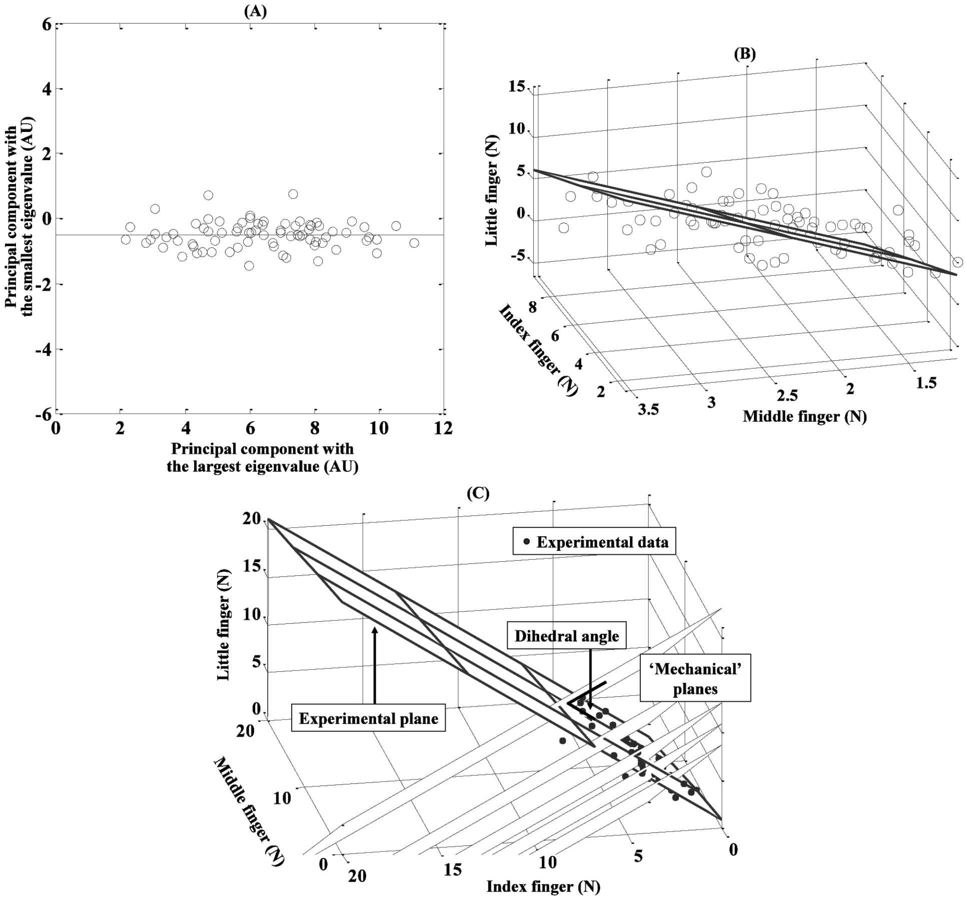


Figure 3 — The experimental data in various projections. (A) The scatter plot of finger forces projected onto the principal components with the largest (abscissa) and smallest (ordinate) eigenvalues. Data points from Subject 1. (B) The finger force distribution in 80 trials and the hyperplane containing experimental data (experimental hyperplane) projected onto a three-dimensional space. Data points from Subject 1. (C) The same hyperplane as in Figure 3A with five "mechanical" planes added. The latter planes are randomly picked up from 80 planes representing the trials with different mechanical constraints and projected onto the three-dimensional space. Each of the "mechanical" planes is plotted by using the following procedure: (1) 100 solutions satisfying each constraint set (each trial's mechanical requirements) are computed; (2) index, middle and little finger's forces from the 100 solutions are picked out and the principal component analysis (PCA) is performed on them; (3) the lesser principal component obtained from the PCA is used to plot the plane in three dimensions. The angle between the experimental plane and the "mechanical" planes in the projected space (89.8°) is different from the real angle in the four-dimensional space (79.5°) due to the projection. Note: the observed data points, their principal components and experimental plane, are all obtained in a 4-dimensional space of finger forces . To visualize the data, we projected them in 2-D and three-dimensional spaces, respectively. The projection of the 4-D experimental hyperplane and observed data onto a three-dimensional space could cause visual distortion of their real orientation and relative distances. The further explanation is provided in the text (see Section 2. Problem Specification).

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ Y_{in} & Y_{mi} & Y_{ri} & Y_{li} \end{pmatrix}$$

$$B = \begin{pmatrix} F^n_{total} \\ M^n \end{pmatrix} \tag{3}$$

The linear constraints provided two equality requirements on normal force F^n_{total} and moment of normal force M^n.

The 20 trials of natural grasping force exertion for each subject were not included in the inverse optimization procedures, since the subjects were free to choose arbitrary force magnitude instead of the prescribed force shown in equation (2); therefore, only 80 trials (80 points in the 4-D finger force space) from each subject were used to compute the unknown cost function

Implication of Planarity of Observations in the Optimization Modeling

For a system of two linear equations with four unknowns (equation (2) above), the solutions should evidently lie on a 2-dimensional hyperplane in the 4-dimensional finger-force space (provided that F^n_{total} and M^n in the right hand side of the equation (2) do not vary). Such a plane, named "mechanical" hyperplane, is defined by the "purely mechanical" constraints with the specific values of F^n_{total} and M^n. This definition makes "mechanical" hyperplane identical to the uncontrolled manifold (UCM) as defined within the UCM hypothesis (Scholz, Schöner 1999; reviewed in Latash et al. 2007).

In the present experiment the constraints were different in different trials. As a result, every experimentally obtained data point (finger normal forces) corresponded to a particular combination of F^n_{total} and M^n, and lied on its own "mechanical" hyperplane. There should be as many "mechanical" hyperplanes as the F^n_{total} and M^n combinations employed in the experiment, i.e., 80 "mechanical" hyperplanes. In case the points of finger force applications on the sensors do not change across the trials, i.e., Y in equation (2) is the same across trials, the "mechanical" hyperplanes would be parallel to each other (Figure 3C presented five exemplary parallel "mechanical" hyperplanes in projection into a three-dimensional space).

The "mechanical" hyperplanes represent only the mechanical constraints of the tasks. They do not reflect the optimality of actual task performance. The requirement of optimality imposes additional constraints on the performance, and hence on the distribution of the experimental recordings. If a point (solution) were randomly picked up from each "mechanical" hyperplane, the 80 points would form a four-dimensional "cloud" instead of being obliged to be on a certain low-dimensional hyperplane. However, in the current study it was found that the experimental points were confined to a plane (for an example see Figure 3A and 3B) and the plane was different from any "mechanical" hyperplane (Figure 3C). In this experiment, the dihedral angle between experimental hyperplane and "mechanical" hyperplane equaled across subjects 83.7° ± 4.5° (standard deviation). The planarity of the experimental observations for the whole set of trials is not a direct consequence of the task mechanics. It is a nontrivial finding indicating that the finger forces are specified in such a way due to certain motor control mechanisms. Most probably these mechanisms are based on optimality principles. The orientation of the two-dimensional hyperplane containing the experimental observations (the experimental plane) might be significantly different from the one of any of the "mechanical" hyperplanes derived from equation (2).

In the next session, the planarity of the observations is used to compute the unknown cost function.

ANIO Computational Procedures

The computational procedure to find an optimization cost function involves several steps.

Step 1. Identify whether the optimization problem is splittable or not by observing the (4 × 4) matrix $\breve{C} = I - C^T\left(CC^T\right)^{-1}C$. If $\breve{C}$ is block-diagonal, or can be made block diagonal by identical reordering its columns and rows, the problem is splittable and should be divided into subproblems each of which is nonsplittable and can be solved individually.

In the experiment of multifinger prehension, the $\breve{C}$ matrices were computed and the optimization problem was found to be nonsplittable in all cases.

Step 2. Determine the observed hyperplane mathematically. The hyperplane can be defined as

$$AF^n = b \tag{4a}$$

where A is a 2×4 matrix composed of the transposed vectors of the two lesser principal components obtained from the PCA; b is a 2-dimensional vector defined as

$$b = A\bar{F}^n \,, \tag{4b}$$

and $\bar{F}^n = \left[\bar{F}^n_{in}, \bar{F}^n_{mi}, \bar{F}^n_{ri}, \bar{F}^n_{li}\right]^T$ is the vector of the average finger normal forces (the horizontal lines above the symbols signify averaging). Note that the hyperplane determined from equation (4a) is prone to experimental errors ("noise").

As it was mentioned above (Section 2.2), the hyperplanes containing the data points recorded in the experiment were determined (see Figure 3).

Step 3. The goal of this step is to compare the experimentally determined hyperplane derived by using equation (4a) with the theoretical plane derived from the Uniqueness Theorem. According to the Theorem, if there are functions $f_{in}\left(F^n_{in}\right)$, $f_{mi}\left(F^n_{mi}\right)$, $f_{ri}\left(F^n_{ri}\right)$ and $f_{li}\left(F^n_{li}\right)$ satisfying $\breve{C}f'\left(F^n\right) = 0$, then the objective function $J\left(F^n\right) = f_{in}\left(F^n_{in}\right) + f_{mi}\left(F^n_{mi}\right) + f_{ri}\left(F^n_{ri}\right) + f_{li}\left(F^n_{li}\right)$ is essentially similar to the true one up to linear terms (prime symbol denotes a derivative of a function f_i with respect to its argument F^n_i, where $i = in$, mi, ri and li).

The objective function J in the current experiment can be determined based on the following consideration: the experimental data points tend to form a two-dimensional hyperplane $AF^n=b$ while all the experimental observations should also comply with $\breve{C}f'\left(F^n\right) = 0$. To satisfy both these requirements, $f'\left(F^n\right)$ should be a linear function of F^n. As a result, the following formulation can be obtained

$$f_i'\left(F^n_i\right) = k_i F^n_i + w_i \tag{5a}$$

Therefore

$$f_i\left(F_i^n\right)=\frac{k_i}{2}\left(F_i^n\right)^2+w_iF_i^n \tag{5b}$$

where $i = in$, mi, ri and li.

An optimal solution would be expected to define a plane $\breve{C}f'\left(F^n\right)=0$ in the four-dimensional space.

The Uniqueness Theorem predicts that if:

(a) matrix C were precisely known and the moments arms of the finger forces were not varying from trial to trial (the CFA's and d_i were constant),

(b) the subject's performance were perfect, i.e., the equilibrium were ideally maintained (no tremor or minute tilting of the object) and the exerted forces satisfied constraints (2) without a slightest deviation, and

(c) the central controllers of the subjects were strictly fulfilling the optimization function, then the hyperplanes of the experimental data, i.e., the plane defined by equation (4a), and the plane of optimal solutions $\breve{C}f'\left(F^n\right)=0$ would coincide.

We cannot expect, however, actual performance to be ideal. Hence, a given experimental plane should be considered an approximation of the (unknown) plane of perfect performance. The latter plane (the plane of optimal solutions $\breve{C}f'\left(F^n\right)=0$) corresponds to a minimal dihedral angle α with respect to the estimate of the experimental plane. To find the angle the average values predicted from the ANIO method were forced to coincide with the average experimental data.

Step 4. The values of k_i can be numerically computed by minimizing the dihedral angle between the two above mentioned planes: (i) the plane of optimal solutions $\breve{C}f'\left(F^n\right)=\breve{C}\left(KF^n+W\right)=0$ and (ii) the plane of experimental observations $AF^n=b$ with $k_{in}=1$ for normalization in the optimization procedure, where $K=\mathrm{diag}(k_{in}, k_{mi}, k_{ri}, k_{li})$ is a 4×4 diagonal matrix with k_i on the main diagonal; the vector $W=[w_{in}, w_{mi}, w_{ri}, w_{li}]^T$ is chosen to have minimal vector length. Thus we get $W=-\breve{C}KF^n$.

In the case of multifinger prehension, minimization of angle α was achieved numerically by selecting proper values of k_i (for normalization $k_{in}=1$ was selected). The minimal values of α were equal to $2.66\pm0.74°$ for the individual subjects and it was $2.81°$ for the lumped group data. Hence, the estimated experimental plane and the plane of optimal solutions were close to each other.

Step 5. The desired objective function is:

$$\hat{J}=\sum_{i=1}^{n}g_i\left(x_i\right) \tag{6a}$$

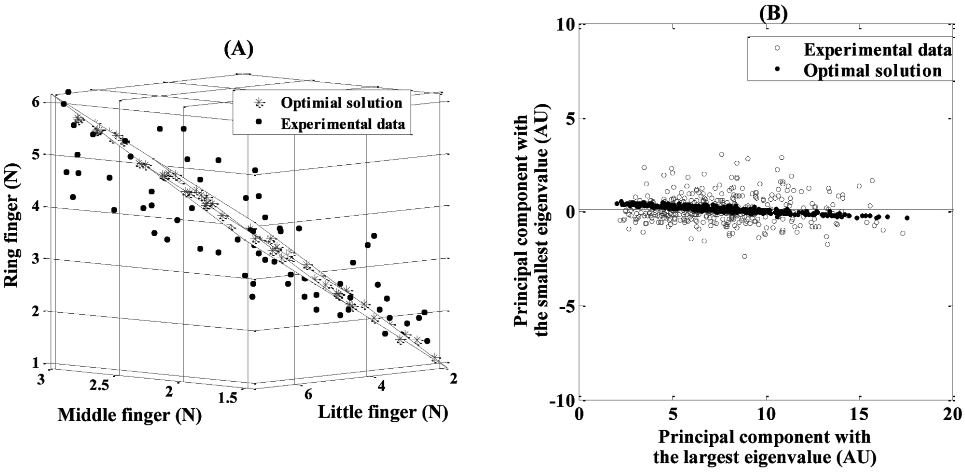
where $g_i\left(x_i\right)=rf_i\left(F_i^n\right)+q_iF_i^n+const_i$, r is a nonzero number, $const_i$ can be any real number, q_i is any real number satisfying the equation $\breve{C}q=0$, where $q=(q_{in}, q_{mi}, q_{ri}, q_{li})^T$ (see detailed explanation in Terekhov et al., 2010). After combining equations (6a) and (5b) and arbitrarily assuming $r=1$, $const_i=0$, and $q_i=0$ we obtain:

$$\hat{J}=\frac{1}{2}\sum_{i=1}^{4}k_i\left(F_i^n\right)^2+\sum_{i=1}^{4}\left(w_i\right)F_i^n \tag{6b}$$

This step completes the computation of the optimization cost functions from the experimental data.

Step 6. To validate the reconstructed objective functions, optimal solutions for different values of vector *B* (equation 3) can be computed and compared with the observations: the solutions should be all lying on the plane $\breve{C}\left(KF^{n} + W\right) = 0$.

In the task of multifinger prehension, this step involves validation of the obtained cost functions by performing direct optimization of finger forces. Though the ANIO method rests on a solid mathematical theory, it did not consider effects of the experimental noise. In the present experiments, the experimental data did not form an ideal plane but instead were scattered around such (see e.g., Panel A in Figure 3 and Panel B in Figure 4). The validation allows ensuring robustness of the ANIO with respect to the experimental noise. The validation was done for the individual subjects as well as for the group data.



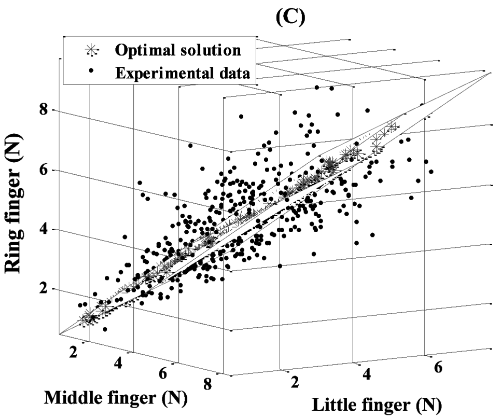


Figure 4 — Projections of the two-dimensional plane of finger forces. (A) Projection onto the three-dimensional space: Middle finger –Ring finger –Little finger. (B) The scatter plot of finger forces projected onto the principal components with the largest (abscissa) and smallest (ordinate) eigenvalues. The small angle between the experimental data and optimal solutions was due to approximation discrepancy between the experimental plane and the plane of optimal solutions, a dihedral angle of 2.81°. (C) Projection onto the three-dimensional space: Middle finger –Ring finger –Little finger. Dots represent the observed experimental data; stars represent the optimal solution obtained from the optimization procedure. Panel A is for Subject 1. Panels B and C are for all subjects.

Validation of the Objective Functions from ANIO and Comparison of Their Performance with Other Cost Functions

The Optimization Results of ANIO

The objective functions were reconstructed from the experimental data following the sequence of steps described above. The procedures were applied to the data points from individual subjects as well as to the pool of data points from all subjects. Original computer codes were written in Matlab software package (Mathworks, Inc., Natick, MA, USA). The optimal solutions were computed by applying the constrained nonlinear multivariate function "fmincon" from Matlab's optimization tool box.

Individual Subjects. The experimentally determined coefficients of Equation 6b are presented in Table 1. All the functions had a similar form across subjects, in particular the coefficients of the second-order terms k_i were positive for all subjects and the relations $k_{in}<k_{mi}=k_{ri}<k_{li}$ were maintained, which was proved statistically.

Table 1 The Estimation of Parameters *ki* and *wi* from the ANIO Method

Subject #	k_{in}	k_{mi}	k_{ri}	k_{li}	w_{in}	w_{mi}	w_{ri}	w_{li}
1	1	2.63	2.23	6.92	-2.34	1.88	3.37	-2.91
2	1	1.59	1.82	2.73	-0.32	0.48	0.10	-0.26
3	1	1.31	1.38	4.46	-1.70	1.21	2.25	-1.76
4	1	1.87	3.09	5.23	-2.47	2.96	1.37	-1.86
5	1	0.85	1.33	1.51	-0.71	1.34	-0.24	0.38
6	1	2.68	1.30	2.09	-0.72	-0.22	2.10	-1.15
7	1	1.42	1.18	2.17	-0.11	-0.22	0.82	-0.48
8	1	1.87	1.83	2.40	-0.11	0.28	-0.21	0.03

The averages of the second-order term coefficient over all subjects were 1.78 ± 0.24, 1.77 ± 0.24, and 3.44 ± 0.72 (average ± *SE*) for the middle, ring, and little fingers, respectively. Paired *t* test showed that k_{mi} was significantly larger than 1 (the normalized value for k_{in}) with $p = .005$; k_{mi} and k_{ri} were the same ($p = .978$) but both smaller than k_{li} ($p < .05$). w_{in} was significantly smaller than w_{mi} and w_{ri} ($p = .012$ and 0.001) but it was the same as w_{li} ($p = .78$); w_{mi} is the same as w_{ri} ($p = .67$). To estimate how different were the $\breve{C}KF = 0$ planes for individual subjects, we computed the maximum angle between the planes for subject 1 and other subjects, and then the maximum angle between subject 2 and others, and so on. The average (± *SE*) of the maximum pairwise dihedral angle between the two hyperplanes was 15.83°±1.59°

The absolute errors between the predicted normal force and the measurement were reasonably small, 0.30 ± 0.01, 0.41 ± 0.02, 0.40 ± 0.02, 0.30 ± 0.01 N across subjects for the index, middle, ring and little fingers, respectively. Figure 4A illustrates the correspondence between the experimental data and the data obtained from the inverse optimization method for one of the subjects.

Since the natural grasping force varied among subjects and torque/load conditions, the finger normal forces expressed in percent of the virtual finger (VF) normal force and their prediction were normalized by the grasping force and the root mean square (RMS) differences between them were computed. The differences were small (Table 2).

Table 2 The RMS Differences Between the Observed Normal Force Sharing Percentages and The Values Predicted from Optimization (In % of The Total Normal Force).

Subject #	1	2	3	4	5	6	7	8	Average ± *SE*
Index finger	1.62	1.25	1.29	2.20	2.07	2.00	2.06	2.29	1.85 ± 0.15
Middle finger	2.47	2.02	1.70	2.89	2.49	2.16	2.82	2.69	2.41 ± 0.16
Ring finger	3.53	1.58	1.47	2.33	2.67	3.12	2.30	2.67	2.46 ± 0.27
Little finger	2.17	1.25	1.10	1.85	2.14	2.03	2.08	2.12	1.84 ± 0.16
Average ± *SE*	0.25 ± 0.46	1.53 ± 0.21	1.39 ± 0.15	2.32 ± 0.25	2.34 ± 0.16	2.33 ± 0.31	2.32 ± 0.20	2.44 ± 0.16	2.14 ± 0.10

The data averaged across eight subjects are presented with standard errors.

Group Data. The angle between the plane $\breve{C}KF = 0$ and the principal components plane was 2.81°. The computed plane $\breve{C}KF = 0$ contained 87.96% of the raw data variance. This result suggested that it was possible to find an estimate of a unified objective function that could fit all eight participants in this experiment. The obtained cost function was:

$$J = \frac{1}{2}\left(\left(F_{in}^{n}\right)^{2} + 1.68\left(F_{mi}^{n}\right)^{2} + 1.83\left(F_{ri}^{n}\right)^{2} + 3.14\left(F_{li}^{n}\right)^{2}\right) - 0.95F_{in}^{n} + 1.06F_{mi}^{n} + 0.77F_{ri}^{n} - 0.87F_{li}^{n} \tag{7}$$

The coefficients of the second order terms were close to the averaged coefficients of the objective functions from the individual subjects.

The data points from all subjects also dispersed around a two-dimensional plane in the four-dimensional space of the fingers' normal forces (Figure 4C). The relations between the experimental data and the data predicted by the optimization algorithm are presented in Figure 5.

The absolute errors between the predicted normal forces and the measurements were 0.49 ± 0.02, 0.67 ± 0.02, 0.57 ± 0.02, 0.46 ± 0.02 N for the index, middle, ring and little fingers, respectively. The maximum absolute error was 2.08, 2.77,

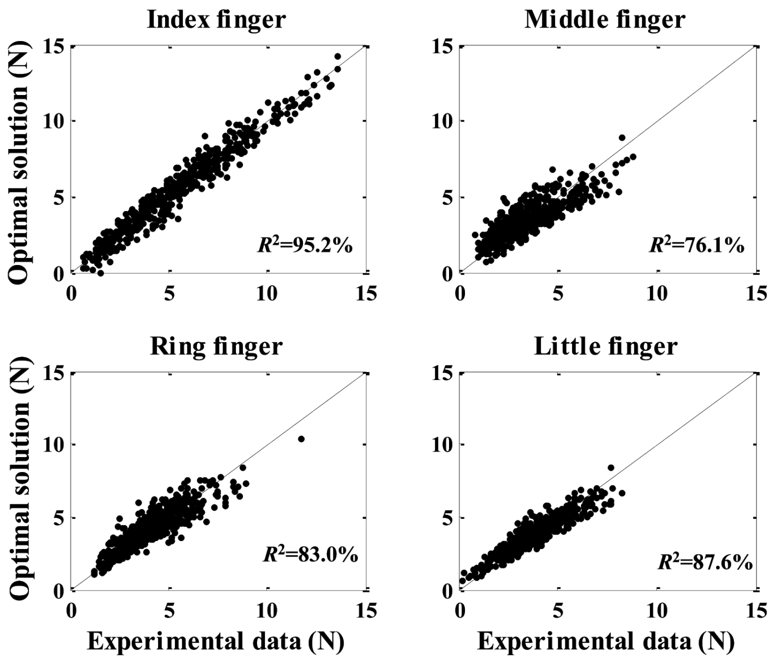


Figure 5 — The scatter plots of the experimental data and the optimal solution for all eight subjects. (A) index finger; (B) middle finger; (C) ring finger; (D) little finger.

2.53 and 1.90 N for each finger across all trials. The RMS differences between the observed normal force sharing percentages and the prediction from optimization were 3.81%+0.46%, 4.81%+0.46%, 4.57%+0.57% and 3.7%+0.38% for the index, middle, ring and little fingers, respectively.

Comparing the ANIO Performance with the Performance of Other Cost Functions

So far, to the best of our knowledge only one method for inferring from experimental data the optimization cost functions has been suggested in the literature (Bottasso et al. 2006). The method was developed for the multibody models of the neuromusculoskeletal system (Multibody Inverse Optimization method, MBIO method). The method has not been validated for multifinger prehension. The method involves limiting the search space for the cost function based on the researcher's knowledge and intuition and then solving a nested optimization problem, i.e., finding a cost function that best matches the experimental data. The (unknown) cost function is assumed to lie in a search space that depends on a set of parameters *p,* i.e., the cost function belongs to a known parameterized class of objective functions. When testing the method we tried quadratic, cubic, and quadric polynomials as the objective functions but for the cubic and quadric polynomials the optimization procedure failed to converge to the experimental data. Therefore we limited the choice to the quadratic functions and used the following polynomial form S_J of finger normal forces without interaction terms as the possible cost function:

$$S_J\left(F_k \mid P\right) = p_{in,1}F^n_{in,k} + p_{mi,1}F^n_{mi,k} + p_{ri,1}F^n_{ri,k} + p_{li,1}F^n_{li,k} \\ + p_{in,2}\left(F^n_{in,k}\right)^2 + p_{mi,2}\left(F^n_{mi,k}\right)^2 + p_{ri,2}\left(F^n_{ri,k}\right)^2 + p_{li,2}\left(F^n_{li,k}\right)^2 \tag{8}$$

where $P = \left(p_{in,2}, p_{mi,2}, p_{ri,2}, p_{li,2}, p_{in,1}, p_{mi,1}, p_{ri,1}, p_{li,1}\right)^T$ is an unknown (8 × 1) vector of coefficients in the objective function S_J; the first subscript of p denotes individual fingers (*in, mi, ri, li*); and the second subscript denotes the order of the terms (1 is for the first-order term and 2 for the second-order term). Note that cost function (8) is of the same class as equation (6b). The difference is that cost function (6b) was obtained from experimental data in the ANIO procedure while quadratic polynomial (8) was suggested by the researchers.

The goal of the MBIO method is to find the best estimate of the coefficients P in the polynomial objective function S_J, i.e., a set of coefficients that minimizes the error (Euclidean distance) between the vectors of experimental observations and optimal solutions. The MBIO method involves breaking the entire optimization procedure into two complementing optimization subroutines: outer and inner optimizations. The inner objective function minimizes polynomial S_J with respect to the normal finger forces. The outer optimization searches the space of the coefficients of S_J to minimize the discrepancy between the solutions of the inner optimization and the experimental observations. The structure of the optimization procedure is hierarchical: the optimal solutions generated by the inner subroutine are used by the outer.

The problem can be reformulated as the following.

Let Z: $R^n \to R^1$, J: $R^n \to R^1$

Outer optimization problem:

$$\min_P Z\left(F^n\right) = \frac{\sum_{k=1}^{N}\left\|F^n_k - \left(F^n_k\right)^E\right\|}{\sum_{k=1}^{N}\left\|\left(F^n_i\right)^E\right\|} \tag{9}$$

subject to $p_{in,2} = 1$, where F^n_k is the solution of the inner optimization problem.

Inner optimization problem:

$$\min_F J = \sum_{k=1}^{N} S_J\left(F^n_k \mid P\right) \tag{10}$$

subject to:

$$\begin{cases} F^n_{in,k} + F^n_{mi,k} + F^n_{ri,k} + F^n_{li,k} = F^n_{total,k}/2 \\ F^n_{in,k}Y_{in,k} + F^n_{mi,k}Y_{mi,k} + F^n_{ri,k}Y_{ri,k} + F^n_{li,k}Y_{li,k} = M^n_k \end{cases}$$

where k =1, . . . 80 is the trial number; $F^n_k = \left(F_{in,k}, F_{mi,k}, F_{ri,k}, F_{li,k}\right)^T$ is a (4 × 1) vector of normal finger forces in trial k; $F^n_{total,k}$ is the virtual finger normal force in trial k; M^n_k is the moment of normal force in trial k; superscript E denotes experimental observations; $h(P)$ is the constraint on vector P that normalizes the coefficient $p_{in,2}$ of the $F^n_{in,k}$ to be 1. J indicates summation of S_J over 80 trials.

The nested optimization problem described above is difficult to solve directly by general numerical methods. To simplify the solution, Bottasso et al. (2006) proposed transforming the inner optimization problem into its dual problem, the Lagrange equation. As a result, the newly generated Lagrange equation together with *h(P)* worked as constraints for the objective function Z. The MBIO method requires huge computational work; for individual subjects the optimization should be performed in a 488-dimensional space (8 polynomial coefficients + 80 trials×4 finger forces +80 trials × 2 slack variables in the Lagrange's functions =488).

We compared the ANIO performance with the MBIO performance (using the software codes that we wrote) as well as with the performance of three other cost functions that have been used previously in the studies on optimization of the finger forces in multifinger prehension tasks (Zatsiorsky et al. 2002b, Pataky et al. 2004b, Niu et al. 2009).

1. Energy-like function over F^n.

$$F_1 = \sum_{i=1}^{4} \left(F_i^n\right)^2 \tag{11a}$$

2. Cubic norm function over F^n.

$$F_2 = \sqrt[3]{\sum_{i=1}^{4} \left(F_i^n\right)^3} \tag{11b}$$

3. Entropy-like function F_3. The function, firstly introduced in robotics (Hershkovitz et al. 1995, 1997), resembles the entropy function in information theory and is minimal when the grasping force is evenly distributed among the fingers.

$$F_3 = \sum_{i=1}^{4} \left(F_i^n + 1\right) \log_{10} \left(F_i^n + 1\right) - \sum_{i=1}^{4} F_i^n \tag{11c}$$

The optimal solutions for the three latter functions were calculated by using Matlab's *fmincon* routine (Mathworks, Inc., Natick, MA, USA).

RM-ANOVAs were performed for each finger with the factors of TORQUE, LOAD, TARGET FORCE and METHOD, where METHOD included five levels corresponding to the procedures of ANIO, MBIO, and another three direct optimization methods. It was found that the effects of TARGET FORCE, LOAD and the interaction METHOD×TORQUE were significant ($p < .0001$ for each). One-sided multiple Bonferroni comparison test with ANIO as the control proved that ANIO method yielded significantly smaller or equal absolute errors to other methods at all TORQUE levels for each finger ($p < .02$ for each).

To further check the performance of the employed optimization method we used three parameterization variables: (a) the dihedral angle between the hyperplanes predicted by a given optimization method and the plane composed by the experimental data; (b) the root mean square (RMS) difference between the experimental data and the data predicted by the optimization procedures, and (c) the average data predicted by the optimization method (with the exception of

the ANIO method, because the average data in this method coincided with the experimental data due to the employed optimization algorithm). For the sake of illustration the obtained results are presented in Table 3. Note that because some

Table 3 Comparing the Performance of the Employed Optimization Methods.

Parameters	Fingers	Experiment	ANIO	MBIO	F_1	F_2	F_3
(a) Dihedral angle	—	—	2.81°	5.24°	21.55°	24.64°	18.25°
(b) RMS Sharing percentages	Index	—	3.8%	4.8%	4.1%	4.8%	3.6%
	Middle	—	4.8%	6.1%	6.2%	7.0%	5.6%
	Ring	—	4.6%	5.3%	3.6%	3.8%	3.6%
	Little	—	3.7%	4.24%.	3.2%	3.9%	2.9%
(c) Average values (N)	Index	5.50	5.50	4.96	4.94	4.82	5.09
	Middle	3.49	3.49	4.13	4.45	4.59	4.32
	Ring	4.30	4.30	4.62	4.09	4.25	3.97
	Little	3.80	3.80	3.38	3.58	3.43	3.70

(a) The angle between the observed data and the optimal solution, (b) the averaged RMS difference between the observed normal force sharing percentages and the values predicted from optimization, and (c) the average values of the experimental observations and the optimal solutions from the ANIO method and the direct optimization (N). ANIO: Analytical inverse optimization; MBIO: Multibody inverse optimization; F_1: Energy-like function; F_2: Cubic norm function; F_3: Entropy-like function

of the above parameters were computed for the entire subject group the statistical testing was inapplicable in this case.

The MBIO method in general worked well yielding the solutions that were close to the ANIO method and the experimental data (Figure 6). However the MBIO accuracy was slightly worse than that for the ANIO method (Table 3). This difference cannot be however attributed to the methods themselves. In MBIO method, we tried 500 different starting points in the solution space based on uniform distribution for each variable, and the solution with the minimal error was selected. Nevertheless, it is still possible that the difference between the performance of the ANIO and MBIO methods was due to an inappropriate selection of the starting point in the MBIO method. Since the numeric optimization was based on the gradient descent method, we can assume that the obtained MBIO solutions correspond to the local minimums, nearest to the starting points. Finding an optimal starting point in a 488-dimensional space is a challenging task requiring huge computations.

The program to compute the objective function by the MBIO method that we developed failed to work with the group pooled data. The size of the vector used in the optimization procedure was too large for solving such a nonlinear problem (the vector size was 3848×1). The method needs further improvement on its numerical computation algorithm.

The optimal solutions that the objective functions F_1, F_2, and F_3 generated were all situated on two-dimensional hyperplanes in the four-dimensional space. However, the dihedral angles between these hyperplanes and the plane of the optimal solution $\breve{C}f'\left(F^n\right)=0$ were large (Table 3). Since the observations and the solutions predicted by the objective functions F_1, F_2, and F_3 were concentrated around their respective averages (see Table 3), the average differences between the observations and optimal solutions caused by the hyperplane deviation were not

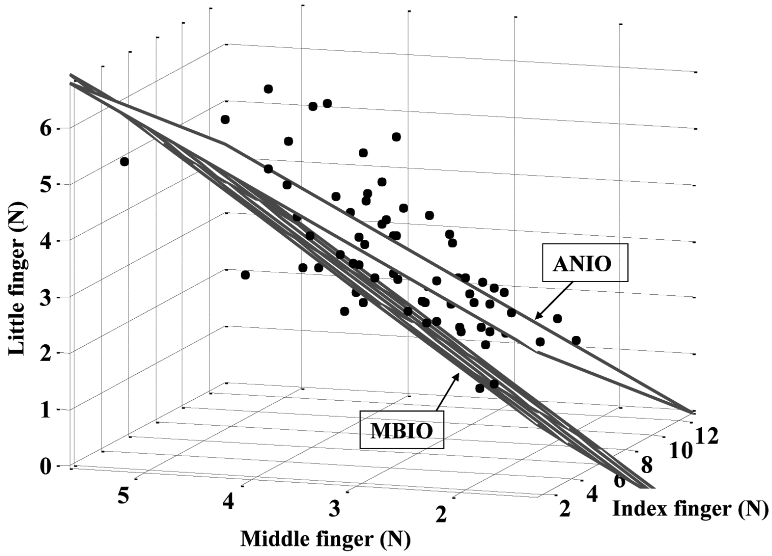


Figure 6 — The observations from Subject 1 and the reconstructed planes from ANIO and MBIO methods projected onto the three-dimensional space: Index finger–Middle finger–Little finger.

large. The absolute errors between the optimal solutions and the observations were small, less than 1 N for each finger. We may expect however that due to the large values of the dihedral angles the errors will greatly increase when the individual data deviate from the group average. However, the sharing percentage errors were all larger than 10%. This discrepancy is due to the relatively small force of natural grasping which results in the small error value but large sharing percentage error when the data are normalized by the total grasping force.

Discussion

Before discussing the goals of the study posed in the Introduction section, we want to briefly address two questions: (A) How was the conclusion that the cost function is quadratic reached (no cost function types were assumed a priori in this research)? (B) How does the ANIO method compare with other methods of inverse optimization?

(A) In brief, the cost function is quadratic because the distribution of the experimental data were (approximately) planar. In other words, the planarity of the distribution of the experimental data demonstrates that the derivatives were linear and hence the cost function was quadratic. If the cost function were not quadratic, e.g., it were cubic, the optimal solutions would not compose a two-dimensional plane, instead they would form a curved surface, e.g., a paraboloid or a hyperboloid in case of a cubic cost function. The reason is straightforward; the derivative of the cubic cost function is quadratic which results in a curved surface for optimal solutions subject to varied constraints.

(B) Other methods of inverse optimization involve, as a rule, assumptions either about a cost function or about its class, e.g., an assumed cost function is quadratic but its coefficients are unknown. Then the parameters of the selected cost function are determined (as we did using the MBIO method) and/or the cost function performance is compared with the performances of other cost functions (as we did in this paper with functions F1, F2, and F3). A notable exception to this practice is a method developed by Körding and Wolpert (2006). The authors proposed an approach to estimating what the subjects considered as "being as accurate as possible" in a particular exemplary task. The reported results are probably the closest to what we did. However, these results do not intersect with our current work. The two methods address similar issues: "measuring the loss associated with error" in the K&W paper and reconstructing the cost function in our study. In both studies, the cost functions were derived from experimental data, and the quadratic cost functions play a special role. Otherwise the studies and the methods are quite different. The K&W study deals with a kinematic task where the end-point accuracy in a single direction was considered a measure of optimality of the performance. The quadratic cost (loss) function was selected before the experiment and then its applicability was tested (the function worked well only for small errors). In our study, a satisfactory accuracy of performance (total force and moment production) was required and was included as a 2-D constraint in the equations. The forces of individual fingers were optimized. Neither a class of the cost functions nor its parameters were assumed. Rather, with an assumption of additive cost function, they were determined (reconstructed) from the experimental data. It was found that an optimal cost function was a second order polynomial with linear terms. Essentially, we studied the famous force sharing problem which has been commonly studied as the problem of sharing muscle forces at a joint (e.g., Seireg, Arvikar 1975; Dul et al. 1984; Herzog 1996; Ait-Haddou et al. 2000, 2002, 2004). In terms of mechanics, Körding and Wolpert (2006) studied kinematics of a serial kinematic chain, while we studied statics of a parallel mechanism (human hand as a parallel mechanism is discussed in Zatsiorsky, Latash 2008 and the differences between the mechanics of the serial and parallel mechanisms are described in Zatsiorsky, 2002, Section 2.3.7). The K&W method cannot be immediately applied for studying the sharing problem (at least, we do not know how to do this). The ANIO method cannot be applied to the K&W problem (at least, we do not know how to do this). Our understanding is that these methods do not overlap but rather complement each other. On the whole, the problem of optimization in motor control is a large problem. It consists of several subfields, which require different methods, such as for instance K&W and ANIO methods.

The further discussion addresses the three goals posed above at the end of the Introduction section.

Does the ANIO Method Work: Is It Possible to Reconstruct the Unknown Cost Function from Experimental Data?

For the studied case of the multifinger prehension the answer is "yes", at least within the simplifications of the experiment and analyses. The validation results presented above in Section 3 support this conclusion. The above answer is evidently limited to the studied prehension task: prismatic vertical grasps with all the points of the digit force application in the grasp plane and the external moment in the plane of grasp (the axis of the moment is orthogonal to this plane). The task mechanics allowed describing it with linear equations and applying the classical methods of linear algebra. Whether this method will work for more complex grasps, for instance the grasps used by the pitchers performing curve ball pitches, cannot be said without additional analysis. Another simplification was limiting the analyses to normal forces only.

The ANIO method is based on the Uniqueness Theorem which proves that the optimal solutions of the additive cost functions should be on the hypersurface, defined by the equation $\breve{C}f'\left(F^n\right) = 0$. The present experiment confirmed that the experimental data lied on a plane or were very close to it, thus showing that there may be an optimization process underlying the production of digit forces. Moreover, the experimental plane was such that it allowed the approximation of the specific type ($\breve{C}f'\left(F^n\right) = 0$), thus proving that the objective function indeed may be additive. Note, that not every surface can be approximated with this kind of equation.

In the PCA, 93.53 ± 0.43% of the total variability was accounted by the two largest eigen values, i.e., the main part of the data variability was along only two eigenvectors. In principle, the planarity of the solution space (the distribution of the experimental data) may be a consequence of a small magnitude of variation of the task constraints in the right hand side of equation (2) across the trials (if this variation were zero the finger force data would be on a hyperplane). To check for such a possibility, we selected a random point based on a uniform distribution from 0 to 15 for each finger force from each 2-dimensional plane (these planes satisfy the mechanical requirements of the tasks, they are "pure" mechanical planes) and performed the PCA on 80 points reconstructed from 80 solution spaces, one for a trial. Such resampling and PCA procedures were repeated 500 times for each subject. The variance explained by the first two principal components varied between 60% and 80% among all subjects' resampling results, which is significantly smaller than the variance from the observations (93.53 ± 0.43%). One-sided *t* test was performed for each subject to compare its variances explained by the first two principal components after the resampling and the variance value observed in the experiment (Figure 7). The observed variances along the first two principal components were larger than the variances generated by the resampling ($p < .001$ for all subjects, see Figure 7A for the boxplot). Hence, the observed variances *along the last two principal components* were smaller, i.e., the experimental data had a tendency to lie closer to a plane than the data obtained from the resampling procedure. The increased planarity of the solution space can be attributed to the intervention of the motor control mechanisms.

The planes of the experimental data and the planes determined from the resampling procedure were quite different (Figure 7 B). Note that all the 500 four-force-value sets obtained from the resampling procedure satisfy the task mechanical requirements. In other words, the value sets located on the plane based on resampled data could solve the task. They are different however from the experimental observations. It seems that the central controller compels the data to lie onto the experimental plane following an optimization procedure, i.e., by minimizing a certain cost function.

The term "reconstruction of the cost functions used by the central controller" should not be taken literally. More accurate expression would be: the cost function

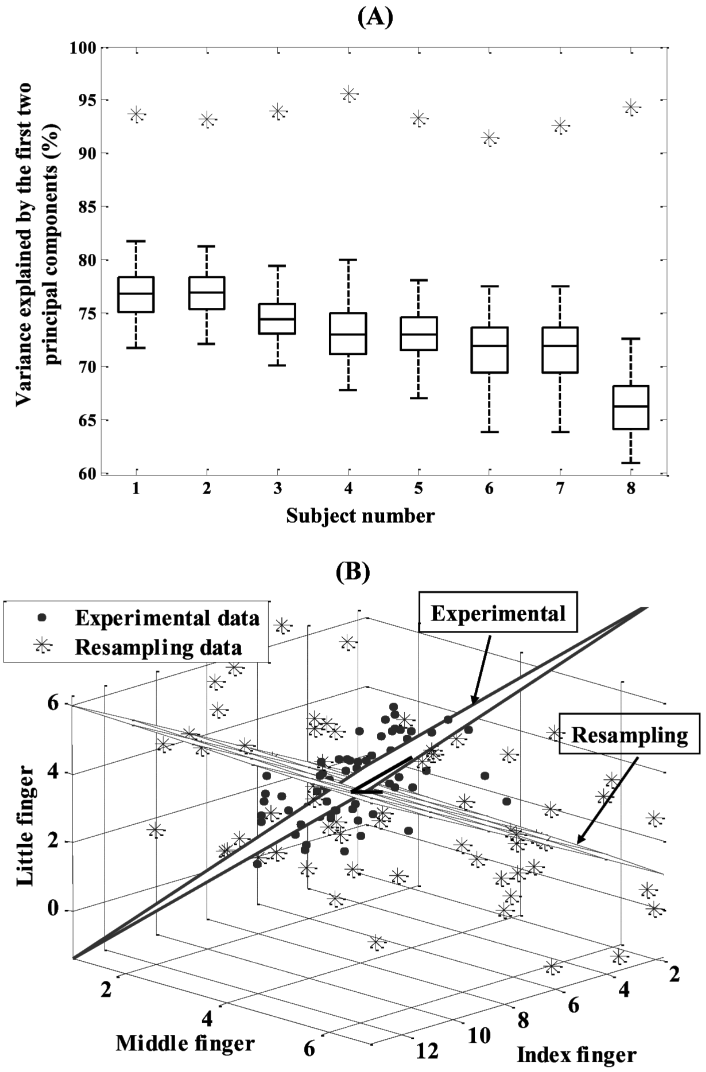


Figure 7 — The illustration of the numerical resampling results: a random point was picked from each hyperplane (corresponding to each trial); 80 points were generated corresponding to the 80 trials for each subject. The PCA was performed on the resampled points. Such a resampling was repeated 500 times for each subject. (A) The boxplot of the variance explained by the first two principal components computed based on the resampled data. Asterisks represent the value of the variance explained by the first two principal components observed in the experiment for each subject. (B) Numerical resampling for subject 1 with the variance of 74.26% explained by the first two principal components. A projection of the 4-dimensional plane onto the 3-dimensional space of the Index-Middle-Little finger forces. The dihedral angle between the observed plane and the resampling plane is shown with thick lines. The angle in this example is 49.3°. The mean value of the angle and its standard deviation are 51.1° and 9.4°, respectively.

reconstructed with the ANIO method approximates (or estimates) a cost function presumably used by the central controller. The ANIO-reconstructed cost function was quadratic. The quadratic form of the objective function is a direct consequence of linear approximation of the experimental data (hyperplane). Since the experimental data were "noisy", it is quite possible that "true" (or ideal) experimental data would form a different hypersurface, for example, a part of a hyperboloid, or even a more extravagant object. For such a surface, the estimated objective function would be different from the quadratic polynomial. Nevertheless, in the same way as the PCA hyperplane can be thought to approximate an ideal hypersurface, the quadratic polynomial approximates the "true" objective function. We evidently do not think that the central controller directly uses quadratic functions (most probably, it does not know algebra); rather, it uses the procedures whose outcome in this particular case can be well approximated by the quadratic polynomials as cost functions.

There are many questions in this field that remain unsolved. An example of such a question is: How do the results of the current study which seem to suggest that a certain optimization function may explain the experimental findings on finger forces in multifinger prehension agree with the previously reported data on the trial-to-trial variability of the forces recorded during repetitive attempts at the same task in seemingly constant conditions (Zatsiorsky et al. 2002a, Shim et al. 2003, 2005). We may hypothesize that the trial-to-trial variability is due to the slightly changed locations of the points of finger force application in different trials, but we cannot either prove or disprove this assertion due to the lack of experimental data. We are going to perform a study in which the same subjects will be tested repeatedly over several days to answer the two questions: First, can the trial-to-trial variability be explained by the different positioning of the fingers in the individual trials? and, Second, do the determined cost functions remain the same for a given subject over several days?

A very intriguing question is whether the obtained cost function indicates that the central controller minimizes a certain mechanical variable whose minimal level corresponds to a minimum of a quadratic objective function or such an outcome is a by-product of minimization of an unknown physiological characteristic rather than a mechanical one. Such a candidate physiological variable can be, for instance, total activity of a set of neurons involved in the control of a motor task. Gelfand and Tsetlin (1966) pointed out that the ensembles of neurons which are not pacemakers, i.e., the cells that need external input to become active, have a natural tendency to go from activity to rest. For the sake of illustration imagine a set of active neurons with only excitatory synapses among the cells that form a closed system, i.e., they do not receive inputs from other parts of the brain and sensory organs. If one cell becomes inactive it ceases to send excitation to other cells. This may be sufficient to deactivate another cell. As a result, a cascade of deactivation occurs and the activity of the entire network decreases or even comes to a close ("the principle of minimization of neural activity", or "minimal final action", Latash 2008, 2010[AUQ4], which is analogous to Hamilton's principle of least action in classical mechanics). Such a "physiological" minimization may be associated with a decrease in many biomechanical variables, such as, for instance, force, power, energy expenditure, etc. As a result of evolution and learning, the "physiological" minimization will be associated with minimal values of some biomechanical variables to a larger degree than with minimization of other variables.

In addition, it is possible that the goal of the optimization is not to minimize an output, e.g., the energy expenditure—after all there is no energy receptors in the body, but to decrease the input, especially unpleasant one, such as for instance the tissue deformation and the afferent signals associated with it.

For the finger tip deformation, it has been shown that finger mechanoreceptor activity is strongly associated with strain energy (Dandekar et al. 2003). Based on this information, Pataky (2005) suggested using for optimization of intrafinger normal-tangential force coordination a quadratic polynomial whose terms are the strain energy of the individual elements in a finite element model. The optimization results agreed well with the experimental observations. The results of McNulty et al. (1999[AUQ5]) who found that fingertip mechanoreceptors generate afferent signals sufficiently strong to activate finger flexor motoneurons also lend credibility to this hypothesis. It is quite possible that the Pataky's hypothesis on strain energy minimization can be extrapolated to multifinger prehension.

Do All Subjects Use Cost Functions of the Same Class, E.G., Quadratic Function with Linear Terms?

Based on the results of the current study we can conclude that the functions are similar, quadratic polynomials with linear terms (see Table 1). The values of k_i coefficients were however different. Presently we do not know whether the sets of k_i coefficients represent a stable trait of individual subjects or they are changeable and vary from one test session to another. We are going to explore this issue in future studies.

Performance of the Cost Functions Determined Analytically Using the ANIO Method In Comparison with Other Cost Functions.

The cost functions reconstructed with the ANIO method worked better than other functions. With respect to the MBIO method, this may be due to an inappropriate selection of the starting point for the numerical optimization. The method assumes that a parameterized class of the objective functions is somehow known. In the current work we used a class of the objective functions (quadratic polynomials with linear terms) yielded by the ANIO method. The MBIO method is supposed to find such parameters of the objective functions that the distance between the experimental data and the theoretical predictions is minimal. However, in the current work we found that the objective function estimated using the ANIO method resulted in smaller errors than the one provided by the MBIO method. Theoretically, this looks impossible, since MBIO is supposed to provide the best possible estimate on the given class. However, we used a method of numeric optimization for the MBIO and it seems that this method converged to a local minimum rather than to the global one, and thus resulted in a set of parameters of the objective function, which are "better than any ones around", but yet are not the best ones. We believe that, if not for the problem of convergence, the MBIO and ANIO methods could form a strong tandem for solving inverse optimization problems: ANIO methods could be used to estimate a parameterized class of the objective functions, while the MBIO method would allow finding the best possible parameters within that class.

The better performance of the ANIO method in comparison with objective functions F_1, F_2, and F_3 is understandable. The F_1, F_2, and F_3 functions are just educated guesses. So, there is nothing strange in the fact that these guesses were not very precise. It so happened that the ANIO method yielded a quadratic function in this experiment. The F_1 function is also quadratic. The terms of the ANIO function however included coefficients k_i, i.e., the function was a weighted function of quadratic values of the finger forces while the terms in F_1 function are not weighted.

The functions inferred from the experimental data, ANIO and MBIO functions, both worked better than the F_1, F_2, and F_3 functions. This result could be expected provided that both ANIO and MBIO methods work. They did. The ANIO yielded more accurate results. It is quite possible that the better performance of the ANIO method will not be seen in other motor tasks.

Still we think that the ANIO method has the following advantages over other methods:

(A) ANIO is a nonparametric method meaning that it does not assume that the objective functions belong to a known parameterized class of objective functions. For real-life applications, the ANIO method can be also used to choose a parameterized class of the objective functions basing on experimental data. In the present work, the fact that the experimental data tended to form a hyperplane in the space of the finger forces dictated the choice of the class of the objective functions: second-order polynomials.

(B) The method is based on the Uniqueness Theorem that provides sufficient conditions for the uniqueness of solutions of the inverse optimization problem with additive objective function and linear constraints. As long as the conditions are satisfied, one can be sure that the method yields a unique solution. Of course, this property holds only for ideal experimental conditions. For real-life applications, it could be very useful to verify that the chosen experimental conditions are sufficient for estimating the objective function. For example, in the current work, estimation of the objective function would be impossible without varying the external torque in the experiment.

We would like to emphasize the uniqueness issue of ANIO method. Of course, given limited and nonideal experimental data it would be naive to expect that the estimated objective function is the "true" one and all other possible functions are false. However, what the ANIO method states is that the "true" objective function used by CNS– assuming that it exists—is sufficiently well approximated by the estimated objective function. Moreover, if someone will ever propose another additive objective function, which would approximate the finger force sharing more precisely than the ANIO method does, this new objective function will be close (on the experimentally observed range of the normal forces) to the quadratic approximation obtained in the current study. It must be noted, that though the mathematical formulation of that hypothetical new objective function could be different from the quadratic polynomial, their "plots" must be close. This statement comes from the theorem of uniqueness and in our opinion represents a major strength of the ANIO method.

Optimization and Motor Control

While optimization approaches have been used in many studies to address problems of motor redundancy, we would like to emphasize their general limitations. First, these approaches are typically formulated in terms of performance variables produced by elements (such as digits, muscles, joints, etc.) of a multielement system. It is well known, however, that muscle activation level and its mechanical output (force, length, their derivatives, and any variables computed from them, such as apparent stiffness and damping) cannot be prescribed by the CNS independently of the external conditions of action execution. This is due to both length and velocity dependence of muscle force and to the action of reflexes (reviewed in Rothwell 1994; Feldman and Latash 2005).

Second, descriptions of optimization approaches frequently create an impression that their authors truly believe that the CNS computes costs of different versions of a planned action and then selects an action with the minimal cost. We would like to distance ourselves from this simplistic understanding. Cost functions, in particular those computed with the ANIO method, may allow to describe behaviors with high accuracy. The fact that such a description is possible and valid across families of tasks is by itself nontrivial. It implies that the functioning of physiological mechanisms involved in the production of natural movements can be described with high accuracy using such methods. The physiological mechanisms themselves are largely unknown; they are likely to involve manipulation of physiological variables that define performance only indirectly, with an important role played by the external force field. It is likely that sets of those variables are also redundant and may be studied using optimization approaches applied to those physiological variables, rather than to outputs of the motor elements of the system (Feldman 2011). It is also possible that laws of physics (physiology) define behaviors without any explicit or implicit optimization, but those behaviors can be accurately described using optimization approaches.

The main results of this study may be interpreted within the framework of a recent hypothesis that combines the ideas of synergic control within a hierarchical system and those of control with referent configurations (Latash 2010a,b). According to the referent configuration hypothesis (which is a daughter of the equilibrium-point hypothesis, Feldman 1966, 1986), neurophysiological control signals lead to setting referent values for important performance variables. This, relatively low-dimensional, task-specific input is mapped on higher-dimensional spaces of body variables (for example, joints, digits, muscles, and motor units) via a hierarchy of few-to-many mappings organized in a synergic way with the help of feedback loops. The last statement means that the "many" output variables covary across trials or along time to produce a stable value of the "few" variables.

At each step of the hierarchy, the low-dimensional input is shared among the higher-dimensional set of elements—a typical problem of redundancy. Theoretically, patterns of this sharing could be defined by pure chance or they could reflect a nonrandom principle based on physical/physiological mechanisms. Optimization approaches search for such nonrandom principles that could be applied across large groups of tasks in variable conditions. A series of recent studies have shown that the ANIO method allows to reconstruct cost functions in young and elderly persons, and in conditions when veridical and nonveridical feedback is provided on performance (Park et al. 2011a[AUQ6],b[AUQ7]). The similarity of the functional forms of the cost functions suggests that they reflect a relatively general principle. On the other hand, lawful changes in the coefficients of such functions in the mentioned comparisons have allowed for offering their biomechanical interpretation. We believe that the application of this method may ultimately help get closer to understanding of physiological mechanisms involved in the coordination of redundant neuromotor systems.

Another important limitation of the ANIO method is that, so far, it has only been used to reconstruct cost functions for static tasks. We hope to develop this method in future to make it applicable for tasks with quickly changing performance variables that are more typical of everyday motor repertoire.

Finally, the scope of motor control problems, to which the ANIO method can be applied is limited by the assumption of linearity of the constraints. Currently, we see only two classes of problems where the constraints can be assumed linear with reasonable precision. The first one concerns the problem of force distribution among muscles in static tasks: as long as the posture is fixed, the muscle force lever arms may be assumed constant and the Jacobian of the mapping between the muscle forces and joint toques is a fixed matrix. The second class includes various problems of finger force sharing in tasks where the fingertip positions are fixed. Similarly, in such tasks, the constraints are linear and the corresponding matrix is fixed.

In a recent review on the optimality principles in sensorimotor control it was mentioned that "It would be very useful to have a general data analysis procedure that infers the cost function given experimental data and a biomechanical model. Some results along these lines are obtained in the computational literature, but a method applicable to motor control is not available" (Todorov 2004, p. 908). The present paper attempts to develop such a method.

Summary

(a) The experimental data recorded from 80 trials with different constraints (load, moment of force, and target grasping force) have a tendency to lie on a 2-dimensional hyperplane in a 4-dimensional finger-force space. Because the constraints in each trial were different such a propensity is a manifestation of motor control mechanisms (not the task mechanics).

(b) The experimental data were found consistent with the hypothesis that the normal force distribution is based on minimization of an additive objective function. This finding allowed for reconstructing the optimization objective functions. The functions happened to be quadratic with linear terms in all subjects.

(c) The estimated objective functions are unique (up to linear members) in a sense, that any additive objective function, explaining the experimental data with the same or higher precision, must be close to the estimated quadratic objective function (up to linear members) or to a function essentially similar to the estimated one.

(d) The central controller not only drives the finger forces to the optimal plane but also selects the latter plane to be different from the planes expected from simple mechanical considerations.

(e) The ANIO method demonstrated good performance in reconstructing the unknown cost functions from experimental data (small RMS errors, small dihedral angle between the planes composed by the observations and optimal solutions, relative simplicity of numerical computations). The method is based on dependable mathematical theory and has an evident potential for examining the inverse optimization problem in motor control.

Acknowledgments

We thank Drs.Y.B. Pesin (Department of Mathematics, The Pennsylvania State University), B. I. Prilutsky (School of Applied Physiology, Georgia Institute of Technology), and S. Piazza (Department of Kinesiology, The Pennsylvania state University) for useful comments on the early version of the manuscript. The authors would also like to thank James Metzler for his technical assistance. The study was partially supported by NIH grants AG-018751, NS-035032, and AR-048563.

References

Abbeel, P., & Ng, A.Y. (2004) Apprenticeship learning via inverse reinforcement learning. Proc ICML: 1-8

Ait-Haddou, R., Binding, P., & Herzog, W. (2000). Theoretical considerations on cocontraction of sets of agonistic and antagonistic muscles. *Journal of Biomechanics, 33*(9), 1105–1011.

Ait-Haddou, R., Jinha, A., & Herzog, W. (2002). In response to "Sensitivity of predicted muscle forces to parameters of the optimization-based human leg model revealed by analytical and numerical analysis" by R.T. Raikova and B.I. Prilutsky and "A physiologically based criterion of muscle force prediction in locomotion" by R.D. Crowninsheild and R.A. Brand. *Journal of Biomechanics, 35*(10), 1433–1435, author reply 1437–1438.

Ait-Haddou, R., Jinha, A., Herzog, W., & Binding, P. (2004). Analysis of the force-sharing problem using an optimization model. *Mathematical Biosciences, 191*(2), 111–122.

Akaike, H. (1974). A new look at the statistical model identification. *IEEE T AUTOMAT CONTR, 19*(6), 716–723.

Aoki, T., Niu, X., Latash, M.L., & Zatsiorsky, V.M. (2006). Effects of friction at the digit-object interface on the digit forces in multi-finger prehension. *Experimental Brain Research, 172*(4), 425–438.

Bottasso, C.L., Prilutsky, B.I., Croce, A., Imberti, E., & Sartirana, S. (2006). A numerical procedure for inferring from experimental data the optimization cost functions using a multibody model of the neuro-musculoskeletal system. *Multibody System Dynamics, 16,* 123–154.

Challis, J.H., & Kerwin, D.G. (1993). An analytical examination of muscle force examinations using optimizations techniques. Proc. Inst. Mech. Eng. Part H J. *Engineering in Medicine, 207,* 139–148.

Crowninshield, R.D., & Brand, R.A. (1981). A physiologically based criterion of muscle force prediction in locomotion. *Journal of Biomechanics, 14*(11), 793–801.

Dandekar, K., Raju, B.I., & Srinivasan, M.A. (2003). 3-D finite-element models of human and monkey fingertips to investigate the mechanics of tactile sense. *Journal of Biomechanical Engineering, 125*(5), 682–691.

Dul, J., Johnson, G.E., Shiavi, R., & Townsend, M.A. (1984). Muscular synergism–II. A minimum-fatigue criterion for load sharing between synergistic muscles. *Journal of Biomechanics, 17*(9), 675–684.

Engelbrecht, S.E. (2001). Minimum Principles in Motor Control. *Journal of Mathematical Psychology, 45*(3), 497–542.

Gelfand, I.M., & Tsetlin, M.L. (1966) On mathematical modeling of the mechanisms of the central nervous system. In: Gelfand, I.M. Gurfinkel, V.S. Fomin, S.V. and Tsetlin. M.L. (Eds.), Models of the Structural-Functional Organization of Certain Biological Systems. Nauka, Moscow, pp.9-26. (in Russian, an English translation is available in 1971 edition by MIT Press, Cambridge)

Feldman, A.G. (1966). Functional tuning of the nervous system with control of movement or maintenance of a steady posture. II. Controllable parameters of the muscle. *Biophysics, 11,* 565–578.

Feldman, A.G. (1986). Once more on the equilibrium-point hypothesis (λ-model) for motor control. *Journal of Motor Behavior, 18,* 17–54.

Feldman, A.G. (2011). Space and time in context of the equilibrium-point theory. *Wiley Interdiscip Rev: Cogn Sci, 2,* 287–304.

Feldman, A.G., & Latash, M.L. (2005). Testing hypotheses and the advancement of science: Recent attempts to falsify the equilibrium-point hypothesis. *Experimental Brain Research, 161,* 91–103.

Hershkovitz, M., Tasch, U., & Teboulle, M. (1995). Toward a formulation of the human grasping quality sense. *Journal of Robotic Systems, 19*(4), 249–256.

Hershkovitz, M., Tasch, U., Teboulle, M., & Tzelgov, J. (1997). Experimental validation of an optimization formulation of the human grasping quality sense. *Journal of Robotic Systems, 14*(11), 753–766.

Herzog, W. (1996). Force-sharing among synergistic muscles: theoretical considerations and experimental approaches. *Exercise and Sport Sciences Reviews, 24,* 173–201.

Herzog, W., & Binding, P. (1993). Cocontraction of pairs of antagonistic muscles: analytical solution for planar static nonlinear optimization approaches. *Mathematical Biosciences, 118*(1), 83–95.

Herzog, W., & Ait-Haddou, R. (2000). Coordination of one- and two-joint muscles during voluntary movement: theoretical and experimental considerations. *Motor Control, 4*(1), 68–74, discussion 97–116.[AUQ8]

Howell, D.C. (2002). *Statistical Methods for Psychology* (5th ed.). Belmont, CA: Duxbury Press.

Körding, K.P., & Wolpert, D.M. (2006). The loss function of sensorimotor learning. *Proceedings of the National Academy of Sciences of the United States of America, 101*(26), 9839–9842.

Latash, M.L. (2008). *Synergy*. New York, NY: Oxford University Press.

Latash, M.L. (2010a). Motor synergies and the equilibrium-point hypothesis. *Motor Control, 14,* 294–322.

Latash, M.L. (2010b). Stages in learning motor synergies: A view based on the equilibrium-point hypothesis. *Human Movement Science, 29,* 642–654.

Latash, M.L., Scholz, J.P., & Schöner, G. (2007). Toward a new theory of motor synergies. *Motor Control, 11,* 276–308.

Littell, R.C., Milliken, G.A., Stroup, W.W., Wolfinger, R.D., & Schabenberger, O. (2006). *SAS for mixed models* (2nd ed.). Cary, NC: SAS Institute Inc.

Liu, C.K., Hertzmann, A., & Popovic, Z. (2005). Learning physics-based motion style with inverse optimization. *ACM Transactions on Graphics, 24,* 1071–1081. [AUQ9]

Ng, A.Y., & Russell, S. (2000) Algorithms for inverse reinforcement learning. Proc ICML: 663-670

Niu, X., Latash, M.L., & Zatsiorsky, V.M. (2009). Effects of grasping force magnitude on the coordination of digit forces in multi-finger prehension. *Experimental Brain Research, 194*(1), 115–129.

Park, J., Zatsiorsky, V.M., & Latash, M.L. (2010). Optimality vs. variability: an example of multi-finger redundant tasks. *Experimental Brain Research, 207,* 119–132.

Park, J., Zatsiorsky, V.M., & Latash, M.L. (in press).Finger coordination under artificial changes in finger strength feedback: A study using analytical inverse optimization. *Journal of Motor Behavior*.[AUQ10]

Park, J., Sun, Y., Zatsiorsky, V.M., & Latash, M.L. (in press). Age-related changes in optimality and motor variability: An example of multi-finger redundant tasks. *Experimental Brain Research*.[AUQ11]

Pataky, T.C., Latash, M.L., & Zatsiorsky, V.M. (2004a). Prehension synergies during nonvertical grasping, I: experimental observations. *Biological Cybernetics, 91*(3), 148–158. [AUQ12]

Pataky, T.C., Latash, M.L., & Zatsiorsky, V.M. (2004b). Prehension synergies during nonvertical grasping, II: Modeling and optimization. *Biological Cybernetics, 91*(4), 231–242.

Pataky, T.C. (2005). Soft tissue strain energy minimization: a candidate control scheme for intra-finger normal-tangential force coordination. *Journal of Biomechanics, 38*(8), 1723–1727.

Prilutsky, B.I. (2000). Muscle coordination: the discussion continues. *Motor Control, 4*(1), 97–116.

Prilutsky, B.I., & Zatsiorsky, V.M. (2002). Optimization-based models of muscle coordination. *Exercise and Sport Sciences Reviews, 30*(1), 32–38.

Rothwell, J.C. (1994). *Control of human voluntary movement* (2nd ed.). London: Chapman & Hall.

Scholz, J.P., & Schöner, G. (1999). The uncontrolled manifold concept: identifying control variables for a functional task. *Experimental Brain Research, 126,* 289–306.

Seireg, A., & Arvikar, R.J. (1975). The prediction of muscular load sharing and joint forces in the lower extremities during walking. *Journal of Biomechanics, 8,* 89–102.

Siemienski, A. (2006). Direct solution of the inverse optimization problem of load sharing between muscles. *Journal of Biomechanics, 39*(Suppl. 1), 45.

Shim, J.K., Latash, M.L., & Zatsiorsky, V.M. (2003). Prehension synergies: trial-to-trial variability and hierarchical organization of stable performance. *Experimental Brain Research, 152*(2), 173–184.

Shim, J.K., Latash, M.L., & Zatsiorsky, V.M. (2005). Prehension synergies: trial-to-trial variability and principle of superposition during static prehension in three dimensions. *Journal of Neurophysiology, 96*(6), 3649–3658.

Syed, U., Bowling, M., & Schapiro, R. (2008) Apprenticeship learning via linear programming. Proc ICML: 1032–1039

Terekhov, A.V., Pesin, Y.B., Niu, X., Latash, M.L., & Zatsiorsky, V.M. (2010). An analytical approach to the problem of inverse optimization with additive objective functions: an application to human prehension. *Journal of Mathematical Biology, 61*(3), 423–453.

Terekhov AV, Zatsiorsky VM (2011). Analytical and numerical analysis of inverse optimization problems: conditions of uniqueness and computational methods. *Biological Cybernetics, 104*(1-2), 75–93.

Todorov, E. (2004). Optimality principles in sensorimotor control. *Nature Neuroscience, 7*(9), 907–915.

Tsirakos, D., Baltzopoulos, V., & Bartlett, R. (1997). Inverse optimization: functional and physiological considerations related to the force-sharing problem. *Critical Reviews in Biomedical Engineering, 25*(4-5), 371–407.

Zatsiorsky, V.M. (2002). *Kinetics of Human Motion*. Champagne, IL: Human Kinetics Publishers.

Zatsiorsky, V.M., Gregory, R.W., & Latash, M.L. (2002a). Force and torque production in static multifinger prehension: biomechanics and control. I. Biomechanics. *Biological Cybernetics, 87*(1), 50–57.

Zatsiorsky, V.M., Gregory, R.W., & Latash, M.L. (2002b). Force and torque production in static multifinger prehension: biomechanics and control. II. *Control. Biol Cybern, 87*(1), 40–49.

Zatsiorsky, V.M., Latash, M.L., Gao, F., & Shim, J.K. (2004). The principle of superposition in human prehension. *Robotica, 22,* 231–234.

Zatsiorsky, V.M., & Latash, M.L. (2008). Multifinger prehension: an overview. *Journal of Motor Behavior, 40*(5), 446–476.

Ziebart, B.D., Maas, A., Bagnell, J.D., & Dey, A.K. (2008) Maximum entropy inverse reinforcement learning. Proc AAAI'08: 1433-1438

Appendix

Uniqueness Theorem

An optimization problem with an additive objective function is defined as:

Let J: $R^n \to R^1$
Min: $J(x) = g_1(x_1) + g_2(x_2) + \cdots + g_n(x_n)$ (12)
Subject to: $CX=B$

where $X=(x_1, x_2, \ldots, x_n) \in R^n$, g_i is an unknown scalar differentiable function with $g'(\cdot)>0$ in the feasible region, C is a $k \times n$ matrix and B is a k-dimension vector, $k<n$. This problem can be recorded as $<J, C>$. Such an objective function is called additive.

Assume that the optimization problem (12) with $k \geq 2$ is nonsplittable. If the functions $g_i(x_i)$ in problem (12) are twice continuously differentiable and there exist twice continuously differentiable functions f_i such that f_i' is not identically constant and $\breve{C}f'(X) = 0$ for all $X \in X^*$, where $f'(X) = \left(f_1'(x_1), \ldots, f_n'(x_n)\right)^T$ and $\breve{C} = I - C^T\left(CC^T\right)^{-1}C$, then $g_i(x_i) = rf_i(x_i) + q_i x_i + const_i$ for every $x_i \in X_i^*$, where $X_i^* = \{s|$ there is $X \in X^*$: $x_i \in s\}$ and X^* is the set of the solutions for all $B \in R^k$, and the constants q_i satisfy the equation $\breve{C}q = 0$ where $q = (q_1, \ldots, q_n)^T$. Primes designate derivatives.

Equation $\breve{C}f'(X) = 0$ ($X \in X^*$) reveals the relation between the objective function and the experimental observations. According to the Lagrange principle, if the experimental data correspond to solutions of an inverse optimization problem with additive objective function and linear constraints then they must satisfy the equation $\breve{C}f'(X) = 0$.The Uniqueness Theorem provides the conditions when the inverse optimization problem can be solved in a unique way (up to linear members) and consequently when the equation $\breve{C}f'(X) = 0$ is sufficient for estimation of the objective function.