Nonlinear control of underactuated vehicles with uncertain position measurements and application to visual servoing

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Abstract— The paper concerns the stabilization of thrustpropelled underactuated vehicles in the context of sensor-based control. First, a class of nonlinear feedback laws that ensure semi-global stabilization despite possibly large uncertainties on the position measurements is proposed. Then, application to the visual servoing of a Vertical Take-Off and Landing (VTOL) drone with a video camera is considered. In particular, it is shown that semi-global stabilization can be achieved, based on homography meaurements, with little information on the environment. Simulation results show the effectiveness of the approach.

I. INTRODUCTION

This paper concerns the stabilization of underactuated thrust-propelled vehicles, i.e. rigid bodies with one bodyfixed thrust control and full torque actuation [1]. Although the present study is not limited to a special class of such systems, it is motivated in the first place by robotic applications with small VTOL UAVs (i.e. Vertical Take-Off and Landing Unmanned Aerial Vehicles). Surveillance of a small geographic zone or inspection of infrastructures (bridges, power lines, etc) are typical examples of such applications. Estimation of the vehicle's pose (i.e. position and orientation) is instrumental in the design of feedback laws that can make the system autonomous. In particular, obtaining a *precise* relative position with respect to the environment is often difficult. This problem is related to the limited payload of small aerial vehicles, which puts severe constraints on the number and quality of embedded sensors. As a consequence, one has to cope with low quality position measurements. Keeping in mind the system's underactuation and its nonlinear dynamics, it is then difficult to guarantee stability of the system in a large domain. The main contribution of this work is the design of a class of feedback controllers that ensure semi-global stabilization of a reference position with possibly large uncertainties on the position measurements (recall that global stabilization cannot be obtained with smooth feedbacks due to topology of SO(3)). One can recast this result in the more general context of sensor-based control [2], [3], where the relation between the measured signal used in the feedback law and the Cartesian coordinates is often poorly known. Sensorbased control is well developed for fully actuated mechanical

systems (like e.g. robotic arms [3]), and some results have also been proposed for nonholonomic wheeled vehicles [4]. The case of underactuated systems, more challenging, has been much less investigated. As a nontrivial application of the proposed result, we address the visual servoing of a UAV in front of a quasi-vertical textured planar structure, based on measurements provided by a video camera.

This work is related to the nonlinear control litterature on VTOL UAVs. Several control design methods have been proposed in order to guarantee semi-global stability¹ (see, e.g., [5], [6], [7], [1]). In those works, however, it is assumed that the position of the vehicle is known. In the present work, we show that such semi-global stability properties can also be ensured despite significant uncertainties on the position vector. The proposed design method builds on the recent work [1] and ideas pertaining to the control in presence of input saturation [8]. It also shares similarities with the method proposed in [9], although the present stability result is stronger as a much larger class of measurement uncertainties is considered. As mentionned above, the present method is also related to sensor-based control, where one tries to assess stability properties based on a rough knowledge of the relation between the sensor output function and cartesian coordinates [3]. To our knowledge, addressing the sensor-based control of underactuated vehicles in a generic way remains to be done. There exist, however, several results on the visionbased control of VTOL UAVs. The application to visual servoing developed in this paper is related to [10], [11], [9], where (semi)-global controllers are also derived based on homography measurements. In particular, uncertainties on the position measurements are also considered in [9]. Due to assumptions on the environment, however, the class of these uncertainties is much smaller than the one here considered. This work is also related to [12] where a nonlinear control approach is proposed for the same visual servoing problem. In that work, however, it is assumed that the normal vector to the planar target, expressed in the body frame, can be extracted from the image. This assumption is not made here. Finally, this result is also related to the authors' work [13] in which homography-based stabilizing controllers were proposed for this problem. Only local stability was proved in that paper, based on the study of the linearized controlled system, whereas semi-global stability is here established. As a counterpart, some assumptions on the target's orientation are needed in this work.

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¹From now on, we use this term to denote asymptotic stability with convergence domain containing all position/velocity initial errors and a neighborhood of the identity matrix in SO(3) for orientation errors.

The paper is organized as follows. Preliminary background and notation are given in Section II. The main result on control design and stability in presence of uncertain position measurements is exposed in Section III. Application to the visual servoing of underactuated UAVs is considered in Section IV and associated simulation results are presented in Section V. Final remarks conclude the paper.

II. PRELIMINARY BACKGROUND

A. Notation

The transpose of a matrix M is denoted as M^T . The $n \times n$ identity matrix is denoted as I_n .

Given a smooth function f defined on an open set of \mathbb{R} , its derivative is denoted as f'.

For any square matrix M, $M_s := \frac{M+M^T}{2}$ and $M_a := \frac{M-M^T}{2}$ respectively denote the symmetric and antisymmetric part of M.

Given $0 < \delta < \Delta$, a function $sat_{\delta,\Delta} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ of class C^1 is called a *saturation function* if:

- i) $sat_{\delta,\Delta}(\tau^2) = 1$ for $0 \le \tau \le \delta$ ii) $\tau sat_{\delta,\Delta}(\tau^2) \le \Delta$ for any $\tau \ge 0$ iii) the function $\tau \longmapsto \tau sat_{\delta,\Delta}(\tau^2)$ is non decreasing on

A function $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ of class C^1 is called a saturating function if:

- *i*) h is strictly positive and bounded on \mathbb{R}^+
- *ii*) $\tau h(\tau^2) < 1$
- *iii*) $\tau h(\tau) \longrightarrow \infty$ when $\tau \longrightarrow \infty$
- *iv*) $h'(\tau) \leq 0$ for any $\tau \geq 0$
- v) the function $\tau \mapsto \tau h'(\tau)$ is bounded on \mathbb{R}^+

Note that both classes of functions play essentially the same role in the control design. Generality considerations have led us to introduce these two classes, but one can consider a unique class composed of functions that are both saturation and saturating functions.

Examples of saturation and saturating function are given by

$$sat_{\delta,\Delta}(\tau) = \begin{cases} 1 \ if \ \tau \le \delta^2 \\ \frac{\Delta}{\sqrt{\tau}} - \frac{(\Delta - \delta)^2}{\sqrt{\tau}(\sqrt{\tau} + \Delta - 2\delta)} \ if \ \tau > \delta^2 \end{cases}$$
$$h(\tau) = \frac{1}{\sqrt{1+\tau}} \tag{1}$$

B. Dynamics of thrust-propelled underactuated vehicles

We consider in this paper the class of thrust-propelled underactuated vehicles consisting of rigid bodies moving in 3d-space under the action of one body-fixed force control and full torque actuation [1]. This class contains most VTOL UAVs (quadrotors, ducted fans, helicopters, etc). The dynamics of these systems is described by the following well-known equations:

$$\begin{cases} \ddot{p} = -uRe_3 + ge_3 \\ \dot{R} = RS(\omega) \\ J\dot{\omega} = J\omega \times \omega + \Gamma \end{cases}$$
(2)

with p the position vector of the vehicle's center of mass, expressed in an reference (inertial) frame, R the rotation matrix from the body frame to the reference frame, ω the angular velocity vector expressed in the body frame, S(.) the matrix-valued function associated with the cross product, i.e. $S(x)y = x \times y, \ \forall x, y \in \mathbb{R}^3, \ u$ the normalized thrust input, i.e. $u = \frac{T}{m}$ where m is the mass and T the thrust input, $e_3 = (0, 0, 1)^T$, J the inertia matrix, Γ the torque vector, and q the gravity constant. We shall consider, by a standard time separation argument commonly used for VTOL UAVs, that the orientation control variable is the angular velocity ω . Indeed, once a desired angular velocity ω^d has been defined, the torque control input Γ is typically computed through a high gain controller:

$$\Gamma = -J\omega \times \omega + kJ\left(\omega^d - \omega\right)$$

with k chosen large enough. Therefore, we focus from now on the subsystem

$$\begin{cases} \dot{p} = \xi \\ \dot{\xi} = -uRe_3 + ge_3 \\ \dot{R} = RS(\omega) \end{cases}$$
(3)

III. CONTROL DESIGN AND STABILITY ANALYSIS

Assume that the following measurements are available:

$$\begin{cases} \pi = R^T M p \\ \gamma = g R^T e_3 \\ v = R^T \dot{p} \\ \omega \end{cases}$$
(4)

with M some 3×3 constant matrix. The measurement π is associated with the vehicle's position, γ is the gravity vector projection in the body frame, v is the linear velocity vector and ω is the angular velocity vector, both expressed in body frame. The matrix R^T in π and γ expresses the fact that these measurements are also obtained in the body frame. This is a typical situation with embedded sensors. In most studies on feedback control of underactuated UAVs, it is assumed that M is the identity matrix, so that the relation between the measurement function and the cartesian coordinates is perfectly known. Several control design methods ensuring semi-global stability of the origin on System (3) have been proposed in this case (see, e.g., [7], [1]). In this section, we show that the same stability properties can be guaranteed in the case of uncertainties on the matrix M, and we provide stability conditions on the control gains in term of the "size" of these uncertainties. In order to provide the rationale of the control design and analysis, we first consider the simpler case of a fully actuated system.

A. The fully actuated case

Consider the position dynamics of a fully actuated rigid body with body fixed force inputs, i.e.

$$\ddot{p} = Ru \tag{5}$$

with $u \in \mathbb{R}^3$ the control input, and assume that the measurements π and v defined above are available for the control design.

Proposition 1 Let $sat_{\delta,\Delta}$ and h denote respectively a saturation and saturating function. Assume that M is positive definite and consider any gain values $k_1, k_2 > 0$ such that

$$\begin{cases} k_2^2 \lambda_{\min}(M_s) > k_1 ||M_a|| ||M|| \sup_{\tau} (h(\tau) + 2\tau |h'(\tau)|) \\ k_2 \delta > k_1 \end{cases}$$
(6)

Then, the feedback law

$$u = -k_1 h(|\pi|^2)\pi - k_2 sat_{\delta,\Delta}(|v|^2) v$$
(7)

ensures the global asymptotic stability and local exponential stability of the origin of System (5).

The proof is given in the appendix.

It follows from the definitions of saturation and saturating functions that the above control law is bounded by $k_1 + k_2\Delta$. Thus, Proposition 1 provides bounded feedback laws that ensure *global* asymptotic stability for a large set of position measurement functions parameterized by M. Furthermore, it provides stability conditions in term of M and the control gains. This can be used to guarantee stability given a priori information on the measurements. Since it is assumed that M is positive definite, it follows from these conditions that stability can always be obtained by choosing k_1 small w.r.t. k_2 . Note also that positivity of k_1, k_2 ensures the stability of the system when M is positive definite and symmetric.

In the following subsection the above result is extended to underactuated systems.

B. The underactuated case

The main result of this paper is stated next.

Theorem 1 Let $sat_{\delta,\Delta}$ and h denote respectively a saturation and saturating function. Assume that M is positive definite and consider any gain values $k_1, k_2 > 0$ such that

$$\begin{cases} k_{2}^{2}\lambda_{min}(M_{s}) > k_{1}||M_{a}||||M||sup_{\tau}(h(\tau) + 2\tau|h'(\tau)|)\\ k_{2}\delta > k_{1}\\ k_{1} + k_{2}\Delta < g \end{cases}$$
(8)

Define a dynamic augmentation:

$$\dot{\eta} = \eta \times \omega - k_3(\eta - \pi), \quad k_3 > 0 \tag{9}$$

together with the control:

$$\begin{cases} \omega_1 = -\frac{k_4 |\bar{\mu}|\bar{\mu}_2}{(|\bar{\mu}|+\bar{\mu}_3)^2} - \frac{1}{|\bar{\mu}|^2} \bar{\mu}^T S(e_1) R^T \dot{\mu} \\ \omega_2 = \frac{k_4 |\bar{\mu}|\bar{\mu}_1}{(|\bar{\mu}|+\bar{\mu}_3)^2} - \frac{1}{|\bar{\mu}|^2} \bar{\mu}^T S(e_2) R^T \dot{\mu} \\ u = \bar{\mu}_3 \end{cases}$$
(10)

where $\bar{\mu}, \mu$, and the feedforward term $R^T \dot{\mu}$ are given by

$$\begin{split} \bar{\mu} &:= \gamma + k_1 h\left(|\eta|^2\right) \eta + k_2 sat_{\delta,\Delta}\left(|v|^2\right) v\\ \mu &:= R\bar{\mu}\\ R^T \dot{\mu} &= -k_1 k_3 \left[h(|\eta|^2) I_3 + 2h'(|\eta|^2) \eta \eta^T\right] (\eta - \pi)\\ + k_2 \left[sat_{\delta,\Delta}(|v|^2) I_3 + 2sat'_{\delta,\Delta}(|v|^2) v v^T\right] (\gamma - u e_3) \end{split}$$

Then,

i) there exists $k_{3,m} > 0$ such that, for any $k_3 > k_{3,m}$, the closed-loop system (3)-(10) together with (9) is asymptotically stable and locally exponentially stable with convergence domain given by $\{\bar{\mu}(0) \neq -|\bar{\mu}(0)|e_3\}$.

ii) if M_s and M_a commute, the same conclusion holds with the first inequality in (8) replaced by:

$$k_{2}^{2}\lambda_{min}(M_{s}) > k_{1}\|M_{a}\|(\|M_{a}\|\sup_{\tau}h(\tau) + \|M_{s}\|\sup_{\tau}2\tau|h'(\tau)|)$$
(11)

The proof is given in the Appendix.

Let us comment on the above result, in relation with the fully-actuated case addressed in Proposition 1. The second and third terms in the definition of $\bar{\mu}$ are reminiscent of the control law (7), with π replaced by η . In view of (9), this latter variable can be viewed as a "filtered value" of π . Note that its time-derivative is known, since it is explicitly defined by (9), while the time derivative of π is not, since M is unknown. This allows to calculate the term $R^T \dot{\mu}$ in (10). Compared to the fully-actuated case, the objective is to make $\bar{\mu}$ converge to $|\bar{\mu}|e_3$ via the definition of u, ω_1 and ω_2 . This ensures that the linear acceleration converges asymptotically to the linear acceleration of the fully-actuated case defined by (5)-(7). Note that ω_3 , which controls the yaw dynamics, is not involved in this objective. Thus, it can be freely chosen. In practice, however, some choices are better than others (see Section IV for an example). Finaly, it follows from (8) that

$$|k_1 h\left(|\eta|^2\right) \eta + k_2 sat_{\delta,\Delta}\left(|v|^2\right) v| \le k_1 + k_2 \Delta < g = |\gamma|$$

This guarantees that $\bar{\mu}(0) \neq -|\bar{\mu}(0)|e_3$ whenever $ge_3^T R(0)e_3 > -(k_1 + k_2\Delta)$ and this ensures, from Property *i*) of the theorem, the semi-global stability of $(\eta, p, \xi, R) = (0, 0, 0, I_3)$. Note that the orientation convergence domain is quite large from an application point of view.

The dynamic extension (9) introduces the complementary gain k_3 with an associated stability condition (an explicit stability condition on k_3 can be deduced from the stability proof but it is complex and probably much conservative). To avoid this extra condition, a simpler control expression with similar robustness properties must be proposed. Although we have not been able to establish a complete stability proof, we conjecture the following result.

Conjecture 1 Suppose that the assumptions of Theorem 1 hold with the following extra assumption on the gains k_1, k_2 :

$$k_1 + k_2 \Delta \le \frac{g}{\sqrt{2}} \tag{12}$$

Define the control law:

$$\begin{cases} \bar{\mu} \triangleq \gamma + k_1 h \left(|\pi|^2 \right) \pi + k_2 sat_{\delta,\Delta} \left(|v|^2 \right) v \\ \omega_1 = -k_4 \bar{\mu}_2 \\ \omega_2 = k_4 \bar{\mu}_1 \\ u = \bar{\mu}_3 \end{cases}$$
(13)

with $k_4 > 0$. Then, the closed-loop system is asymptotically stable and locally exponentially stable and the system's solutions converge to the origin if $e_3^T R(0)e_3 > \frac{k_1+k_2\Delta}{g}$. Moreover, under these conditions, the angle between local and inertial vertical will always remains strictly smaller than $\alpha_0 = \arccos\left(\frac{k_1 + k_2 \Delta}{a}\right).$

We now illustrate Theorem 1 on an example which is instrumental for the visual servoing application studied in Section IV.

Example: Assume that $M = \alpha I_3 + S(\beta)$ with $\alpha > 0$ and S the skew-symmetric matrix associated with the cross product, i.e. $S(\beta)x = \beta \times x$. Then, $M_s = \alpha I_3$ and $M_a = S(\beta)$ commute so that the stability condition (11) applies. Take h and $sat_{\delta,\Delta}$ as defined by (1). The stability conditions on k_1, k_2 are then given by:

$$\begin{cases}
k_1, k_2 > 0 \\
k_2\delta > k_1 \\
k_1 + k_2\Delta < g \\
\alpha k_2^2 > k_1|\beta| \left(|\beta| + \frac{2\alpha}{3\sqrt{3}}\right)
\end{cases}$$
(14)

IV. APPLICATION TO VISUAL SERVOING

In this section, Theorem 1 is applied to a visual servoing application for a VTOL UAV.

A. Preliminary recalls and problem statement

Consider an underactuated UAV with a video camera facing a planar target. Suppose that a "reference" picture of this target taken at a reference pose is known. This reference pose is represented on Fig. 1 by the reference frame \mathcal{R}^* . We assume that this pose is a possible equilibrium for the dynamics of this underactuated vehicle, meaning that the zaxis of \mathcal{R}^* is vertical. Otherwise, asymptotic stablization of this pose would not be possible. Finally, we also assume that the optical center of the camera corresponds to the vehicle's center of mass and the optical axis corresponds to the x-axis of \mathcal{R}^* . At every time, the "current" picture of the target, taken at the current pose represented by the frame \mathcal{R} , is compared to the reference one. From this comparison, the homography matrix is computed (see e.g. [14], [15] for more details on homography matrices and associated computation algorithms). This matrix, which allows to transform the target's points coordinates from the reference pose to the current pose, is given by

$$H = R^T - \frac{1}{d^*} R^T p n^{*T}$$
⁽¹⁵⁾

where d^* is the distance from the UAV reference position to the target plane and n^* is the normal to the target plane expressed in the reference frame. Both variables are unknown since they cannot be extracted from the visual data. Thus, they are not available for the control design. The objective is to design a feedback controller that ensures semi-global stabilization of the vehicle at the reference pose based on these homography measurements together with velocity measurements. To our knowledge, this general problem remains open. In [9], [12], solutions to this problem have been proposed when the normal n^* to the planar target is known. Note that it is then possible to extract from H the rotation matrix and the position vector p up to the unknown scale factor d^* . We address in this section a more complex case for which such an extraction is not possible. More precisely, we consider the case of a vertical target, which is of interest in many inspection applications. This assumption is equivalent to $n_3^* = 0$.



Fig. 1. Problem scheme

B. Error vector definition

In this section, using the assumption of verticality for the planar target, we show that measurements π and γ of the form (4) can be extracted from the homography matrix H. More precisely, let

$$\begin{cases} \pi = He_2 \times He_3 - He_1 \\ \gamma = gHe_3 \end{cases}$$
(16)

From the assumption $n_3^* = 0$, one can verify that:

$$\begin{cases} \pi = R^T M(\frac{n^*}{d^*})p \\ \gamma = gR^T e_3 \end{cases}$$
(17)

with $M(\tau) = \tau_1 I + S(\tau_2 e_3)$.

C. Visual servoing

We assume that velocity measurements v are available. In practice, they can be obtained, e.g., via a GPS². In this case, Theorem 1 applies directly with $M = M\left(\frac{n^*}{d^*}\right)$ and by (14) we deduce that the control law (10) ensures semi-global stabilization of the reference pose provided that:

$$\begin{cases}
n_{1}^{*} > 0 \\
k_{1}, k_{2} > 0 \\
k_{2}\delta > k_{1} \\
k_{1} + k_{2}\Delta < g \\
n_{1}^{*}d^{*}k_{2}^{2} > k_{1}|n_{2}^{*}|\left(|n_{2}^{*}| + \frac{2n_{1}^{*}}{3\sqrt{3}}\right)
\end{cases}$$
(18)

Note that the first condition, which ensures that M is positive definite, essentially means that the camera is "facing" the target at the reference pose. This is a very natural assumption from an application point of view. When (loose) bounds are known for d^* : $d_{\min} \leq d^* \leq d_{\max}$ and $n_1^* \geq n_{\min}$, and

 $^{^{2}\}mathrm{The}$ case of a GPS-denied environment will be considered in a future work.

recalling that $|n^*| = 1$, the last condition of equation (18) can be replaced by:

$$n_{1\min}d_{\min}k_2^2 > k_1\left(1 + \frac{2}{3\sqrt{3}}\right)$$
 (19)

D. Yaw control

The yaw degree of freedom is not involved in the stabilization objective. On the other hand, it matters to keep the target inside the field of view of the camera. We propose to use the following control law:

$$\omega_3 = k_5 H_{21} \tag{20}$$

Upon convergence of the position, velocity, roll and pitch angles due to the other controls, the yaw dynamics will be close to $\dot{\psi} \approx -k_5 \sin\psi$, thus ensuring the convergence of ψ to zero unless ψ is initially equal to π (case contradictory with the visibility assumption). Another nice feature of this yaw control is that it vanishes when $H_{21} = 0$, i.e. when the target is seen -from yaw prospective- as it should be at the end of the control task. This means that the controller tries to reduce the yaw angle only when the position/velocity errors have been significantly reduced. This can be verified on the simulation results presented next.

V. SIMULATION RESULTS

We now illustrate the control approach by simulation results for the visual servoing problem of the previous section. We consider the controller proposed in Theorem 1. The following control parameters are used in all simulations:

- $[k_1, k_2, k_3, k_4, k_5] = [1, 2, 1, 1, 1]$
- the saturation function $sat_{\delta,\Delta}$ and the saturating function h are given by (1) with $\delta = 1, \Delta = 1.1$,

It follows from (19) that for these gain values, the controller should stabilize the UAV for any distance $d^* \geq d_{\min} = 0.49m$ with $n_{1_{min}} \geq 0.7$.

The values for the initial position, velocity, orientations and normal n^* are given by:

• Figure 2: $n^* = (0.7; -0.71; 0)^T$, $d^* = 3m, p_0 = (-5.5m; 1.2m; 1.3m;)^T$, $v_0 = (-2.3m/s; -1.6m/s; 0.4m/s)^T, \phi_0 = -0.5^\circ,$ $\theta_0 = 11.4^\circ, \psi_0 = 4.1^\circ$

• Figure 3: $n^* = (0.8; 0.61; 0)^T$, $d^* = 1m, p_0 = (-6.6m; 5.2m; 3.2m;)^T$, $v_0 = (-1m/s; 0.9m/s; 0.4m/s)^T$, $\phi_0 = 8.1^\circ$, $\theta_0 = 18.3^\circ$, $\psi_0 = -30.7^\circ$

On these figures, it can be seen that the control law stabilizes the system and recovers from large initial conditions.

VI. CONCLUSION

A nonlinear control law has been proposed to semiglobally stabilize underactuated vehicles in the presence of uncertain position measurements. Explicit stability conditions on the control gains, in relation with the measurement uncertainties, have been derived. The approach has been applied to an important visual servoing application and simulation results validate the control design and robustness



Fig. 2. Control law from Theorem 1, test case 1



Fig. 3. Control law from Theorem 1, test case 2

analysis. There are many possible extensions of the present work: extension of the control design and analysis to the more general context of *sensor-based* control, proof of the stability conjecture for the simplified version of the main controller, experimental validation of the visual servoing task here considered, application to other sensor suites, etc.

APPENDIX: PROOF OF PROPOSITION 1

In view of (4), (5), and (7), π and v satisfy the following equations:

$$\begin{cases} \dot{\pi} = \pi \times \omega + R^T M R v \\ \dot{v} = v \times \omega - k_1 h(|\pi|^2) \pi - k_2 sat_{\delta,\Delta}(|v|^2) v \end{cases}$$
(21)

Consider the following change of variables:

$$\begin{cases} x = R\pi \\ y = Rv = \xi \end{cases}$$

Then, System (21) becomes:

$$\begin{cases} \dot{x} = My \\ \dot{y} = -k_1 h\left(|x|^2\right) x - k_2 sat_{\delta,\Delta}\left(|y|^2\right) y \end{cases}$$
(22)

We now focus on the stability analysis of this system.

We first show that, after some time, the y-correction term desaturates, i.e.

$$\exists T/\forall t \ge T, \begin{cases} |y| \le \delta\\ sat_{\delta,\Delta}(|y|^2) = 1 \end{cases}$$

Recall that the two conditions above are equivalent by definition of the *sat* function. Consider the function V defined by $V(y) = \frac{1}{2}|y|^2$ and its derivative along the solutions of System (22). We have:

where the second inequality comes from the assumptions on h. By the definition of a saturation function,

$$|y| \ge \delta \Longrightarrow |y| sat_{\delta,\Delta}(|y|^2) \ge \delta sat_{\delta,\Delta}(\delta^2) \ge \delta$$

Therefore

$$|y| \ge \delta \Longrightarrow \dot{V} \le (k_1 - k_2 \delta)|y| \le (k_1 - k_2 \delta)\sqrt{2V}$$

which is equivalent to

$$V \ge \frac{\delta^2}{2} \Longrightarrow \dot{V} \le (k_1 - k_2 \delta) \sqrt{2V}$$

Since $k_1 - k_2\delta < 0$ by assumption, we deduce that along any solution of System (22), there exists a time T such that $V(y(t)) \leq \delta^2/2$ for $t \geq T$. In other words, $|y(t)| \leq \delta$ for $t \geq T$. Then, $sat_{\delta,\Delta}(|y|^2) = 1$ and the solution is also solution to the following system:

$$\begin{cases} \dot{x} = My\\ \dot{y} = -k_1 h\left(|x|^2\right) x - k_2 y \end{cases}$$
(23)

Let us consider the candidate Lyapunov function V_0 defined by:

$$V_{0}(\kappa, x, y) = k_{1} \int_{0}^{|x|^{2}} h(\tau) d\tau + y^{T} M y + \frac{2k_{1}}{k_{2}} h(|x|^{2}) x^{T} M_{a} y + \kappa h(|x|^{2}) x^{T} y$$
(24)

where κ is a constant positive number to be defined. We show that, for κ small enough, V_0 is indeed a Lyapunov function. We first prove that V_0 is positive definite and proper. Integrating by part and using the fact that $h'(\tau) \leq 0 \,\forall \tau$, we get

$$\int_{0}^{|x|^{2}} h(\tau) d\tau = -\int_{0}^{|x|^{2}} \tau h'(\tau) d\tau + |x|^{2} h(|x|^{2})$$

$$\geq |x|^{2} h(|x|^{2})$$

Thus,

$$\begin{split} V_0(\kappa, x, y) &\geq k_1 |x|^2 h(|x|^2) + \lambda_{\min}(M_s) |y|^2 \\ &- \left(\frac{2k_1}{k_2} \|M_a\| + \kappa\right) \sup_{\tau} \sqrt{h(\tau)} \sqrt{h(|x|^2)} |x| |y| \end{split}$$

and we deduce that V_0 is positive definite provided that

$$\left[\left(\|M_a\|\frac{2k_1}{k_2}+\kappa\right)\sup_{\tau}\sqrt{h(\tau)}\right]^2 < 4k_1\lambda_{\min}(M_s)$$

Since we want this inequality to hold for *some* κ small enough, this condition is equivalent to

$$k_2^2 \lambda_{\min}(M_s) > k_1 ||M_a||^2 \sup_{\tau} h(\tau)$$

With $||M_a|| \leq ||M||$, this condition holds true when the assumptions of Proposition 1 are satisfied. Since h is strictly positive, this ensures that V_0 is a positive definite function of x and y, and V_0 is proper due to the assumption $\tau h(\tau) \rightarrow \tau$

 $+\infty$ as $\tau \longrightarrow +\infty$. Let us now prove that V_0 is nonincreasing along the solutions of System (23). Differentiating V_0 along these solutions yields

$$\begin{split} V_0 &= -2k_2y^T M_s y \\ &- 2\frac{k_1}{k_2}y^T M_a \left[h(|x|^2) + 2h'(|x|^2)xx^T\right] My \\ &- \kappa k_1 h^2(|x|^2)|x|^2 \\ &+ \kappa y^T M \left[h(|x|^2) + 2h'(|x|^2)xx^T\right] y \\ &- \kappa k_2 h(|x|^2)x^T y \end{split}$$

We deduce that

$$\dot{V}_0 \leq -C_1 |y|^2 + C_2 |y| h(|x|^2) |x| - C_3 h^2(|x|^2) |x|^2$$

with:

$$\begin{pmatrix}
C &\triangleq \sup_{\tau} (h(\tau) + 2\tau |h'(\tau)|) \\
C_1 &\triangleq 2k_2 \lambda_{\min}(M_s) - \left(\frac{2k_1}{k_2} \|M_a\| + \kappa\right) \|M\|C \\
C_2 &\triangleq \kappa k_2 \\
C_3 &\triangleq \kappa k_1
\end{pmatrix}$$

We now show that this expression is negative definite for κ small enough. Considering k_1, k_2 chosen, let us first fix an upper bound on κ such that $C_1 > 0$. This is possible since, by assumption

$$\lambda := k_2^2 \lambda_{min}(M_s) - k_1 ||M_a||||M||C > 0$$

Thus, choosing

$$\kappa < \kappa_0 \triangleq \frac{2\lambda}{k_2 \|M\|C}$$

-with the right-hand term being strictly positive- ensures $C_1 > 0$. Now, we add a constraint on the choice of κ by enforcing $C_2^2 < 4C_1C_3$. Again, this is always possible since it boils down to forcing

$$\kappa < \kappa_1 \triangleq \frac{2\lambda}{k_2 \|M\|C + \frac{k_2^2}{4k_1}}$$

Moreover, with $\kappa_1 < \kappa_0$, both inequalities boil down to $\kappa < \kappa_1$. This choice ensures that the derivative of V_0 is negative definite and ends the proof.

APPENDIX: PROOF OF THEOREM 1

The proof shows similarities to the previous one although extra steps have to be considered. First, one easily verifies that π, v, γ , and η satisfy the following equations:

$$\begin{cases} \dot{\pi} = \pi \times \omega + R^T M R v \\ \dot{v} = v \times \omega - k_1 h(|\eta|^2) \eta - k_2 sat_{\delta,\Delta}(|v|^2) v + \epsilon \\ \dot{\gamma} = \gamma \times \omega \\ \dot{\eta} = \eta \times \omega - k_3 (\eta - \pi) \end{cases}$$

$$(25)$$

where $\epsilon = -\bar{\mu}_3 e_3 + \bar{\mu}$.

Let us first prove that $\epsilon \longrightarrow 0$. Recall the following lemma shown in [1]:

Lemma 1 Assume that

- i) $\bar{\mu} = R^T \mu$ does not vanish,
- ii) $\dot{\mu}$ does not depend on ω ,
- *iii*) $\bar{\mu}(0) \neq -|\bar{\mu}(0)|e_3$.

Let k_4 denote a strictly positive constant and

$$\begin{cases} \omega_1 = -\frac{k_4|\bar{\mu}|\bar{\mu}_2}{(|\bar{\mu}|+\bar{\mu}_3)^2} - \frac{1}{|\bar{\mu}|^2}\bar{\mu}^T S(e_1) R^T \dot{\mu} \\ \omega_2 = \frac{k_4|\bar{\mu}|\bar{\mu}_1}{(|\bar{\mu}|+\bar{\mu}_3)^2} - \frac{1}{|\bar{\mu}|^2}\bar{\mu}^T S(e_2) R^T \dot{\mu} \end{cases}$$
(26)

Then $\bar{\mu}$ asymptotically converges to $|\bar{\mu}|e_3$.

The assumptions being obviously verified in our case, the lemma applies and shows that $\epsilon \longrightarrow 0$.

We first analyze System (25) with $\epsilon = 0$. Consider the following change of variables :

$$\begin{cases} x = R\pi \\ y = Rv \\ z = R\eta \end{cases}$$

System (25) becomes:

$$\begin{cases} \dot{x} = My \\ \dot{y} = -k_1 h\left(|z|^2\right) z - k_2 sat_{\delta,\Delta}\left(|y|^2\right) y \\ \dot{z} = -k_3 \left(z - x\right) \end{cases}$$
(27)

As in the proof of Proposition 1, we show that the y term desaturates with the same function V and same argument with z instead of x in the \dot{y} equation. This ensures that for $t \geq T$, $sat_{\delta,\Delta}(|y|^2) = 1$ and the solution is also solution to the system without y saturation, which may be written as follows:

$$\begin{cases} \dot{x} = My \\ \dot{y} = -k_1 h\left(|x|^2\right) x - k_2 y \\ -k_1 \left[h\left(|z|^2\right) z - h\left(|x|^2\right) x\right] \\ \dot{z} - \dot{x} = -k_3 \left(z - x\right) - My \end{cases}$$
(28)

Note that System (28) is equivalent to System (23) when z = x. Asymptotic stability of this system can be proved using the following Lyapunov function:

$$V_1(\kappa, x, y, z) = V_0(\kappa, x, y) + (z - x)^2$$
(29)

Due to space limitation, we only provide a sketch of the proof (details are available upon request to the authors). One can show that:

$$\begin{split} \dot{V}_1 &\leq & -C_1 |y|^2 - C_2 |z - x|^2 - C_3 h^2 (|x|^2) |x|^2 \\ &+ C_4 |y| |z - x| + C_5 |y| h(|x|^2) |x| \\ &+ C_6 |z - x| h(|x|^2) |x| \end{split}$$

with:

$$\begin{pmatrix}
C &\triangleq \sup_{\tau} (h(\tau) + 2\tau | h'(\tau) |) \\
C_1 &\triangleq 2k_2 \lambda_{\min}(M_s) - \left(\frac{2k_1}{k_2} \| M_a \| + \kappa\right) \| M \| C \\
C_2 &\triangleq 2k_3 \\
C_3 &\triangleq \kappa k_1 \\
C_4 &\triangleq 2Ck_1 \| M_s \| + 2 \| M \| \\
C_5 &\triangleq \kappa k_2 \\
C_6 &\triangleq \left(2C \frac{k_1^2}{k_2} \| M_a \| + \kappa k_1 C \right)$$

Chosing κ small enough and k_3 large enough, \dot{V}_1 can be made negative definite. Finally, when a vanishing term ϵ is added to System (28) in the \dot{y} equation, the desaturation argument still holds since we get:

$$\frac{1}{2}|\dot{y}|^2 \le (k_1 - k_2\delta - \epsilon(t))|y|$$

with $\epsilon(t) \longrightarrow 0$ and $k_1 - k_2 \delta < 0$ Also, to the derivative of V_1 is added the term

$$\left(y^T M + \frac{2k_1}{k_2} h\left(|x|^2\right) x^T M_a + \kappa h\left(|x|^2\right) x^T\right) \epsilon(t)$$

This term can be bounded in norm by $c(|y| + h(|x|^2)|x|) |\epsilon(t)|$ with c some constant. As a result, because of the expression of \dot{V}_1 when $\epsilon = 0$, one deduces that $V_1 \longrightarrow 0$ when $t \longrightarrow +\infty$.

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