Feedback control of a redundant wheeled snake mechanism using transverse functions on $\mathbb{SO}(4)$

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Abstract: The Transverse Function (TF) approach is applied to the control of a nonholonomic three-segments/snake-like wheeled mechanism, similar to the planar low-dimensional version of Hirose's Active Cord Mechanism (ACM) previously studied by the authors with the same control approach, but with two additional internal degrees of freedom (d.o.f.) whose actuation yields more flexible and efficient control solutions. From a theoretical point of view, these complementary d.o.f. modify the Control Lie Algebra of the system so that only first-order Lie brackets of the control vector fields are needed to satisfy the Lie Algebra Rank Condition (LARC). The fact that four independent (angular velocity) control inputs are used also implies for this system the existence of Transverse Functions (TF) defined on the six-dimensional special orthogonal group $\mathbb{SO}(4)$. Several examples of mechanisms whose control involve TF defined on $\mathbb{SO}(3)$ have been pointed out in the past. Beyond the specific control problem addressed here, a motivation for the present study is to illustrate for the first time how functions defined on the larger set $\mathbb{SO}(4)$ can be determined and used for the control of a physical system. This study is complemented with recalls concerning the parametrization of $\mathbb{SO}(4)$ by pairs of *isoclinic* quaternions and with the derivation of complementary differential calculus relations associated with this parametrization.

Keywords: snake robot, stabilization, nonholonomic system, transverse function, nonlinear system

1. INTRODUCTION

We are pursuing the development of the Transverse Function (TF) approach, Morin and Samson (2001), Morin and Samson (2003), for the control of highly nonlinear systems. In relation to this endeavour, the study of snake-like wheeled robots gives us the opportunity to i) apply and adapt this approach to various mechanical systems for which no feedback control solution existed so far, ii) prolong and generalize the control design methodology associated with it, and iii) propose new paradigms for the control of systems whose motion capabilities are based on the generation of oscillatory (or undulatory) shape changes.

Of particular interest to us is the control of snake-like wheeled mechanisms, proposed by various researchers to better understand crawling locomotion (starting with the pioneering works of Hirose et al., Hirose (1993), Hirose and Mori (2004)). Indeed, most of the studies devoted to this theme have focused on the generation of open-loop control strategies yielding simple overall displacements along specified (and specific) directions, as in Ostrowski and Burdick (1998), Ishikawa (2009), whereas attempts to synthesize feedback control laws, as in Matsuno and Mogi (2000), are few, incomplete and (to our point of view) mostly inconclusive due to the non-existence of adequate control design tools. One of our objectives is to show that the TF approach, and its extensions, provide such tools.

The first mechanism of this kind that we have considered is the *trident snake* system originally proposed in Ishikawa (2004). This mobile robot has a "parallel" mechanical structure and is composed of a triangular-shaped body with wheeled legs attached at its summits via rotoid articulations. The structure of the Control Lie Algebra associated with the kinematic equations of this system differs from the one of more commonly studied chained systems and gave us the idea to look for new transverse functions defined on the rotation group $\mathbb{SO}(3)$ –instead of the three-dimensional torus $\mathbb{T}^3$–, Ishikawa et al. (2009). The better performance observed in simulation when using these new functions comes from the fact that they better respect the system's symmetries. We have subsequently generalized the construction of such functions on $\mathbb{SO}(n)$ in relation to the case of a Control Lie Algebra maximally generated by Lie brackets of order less or equal to one, Morin and Samson (2009b).

The present study focuses on Hirose's *ACM III snake robot* and, more specifically, on the simplified planar model composed of three segments, as depicted on Figure 1. This

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system can be actuated in various ways. For instance, Ostrowski and Burdick (1998) –a study centered on modeling and open-loop control aspects– considered the case of five control inputs: the velocities of the two articulation angles $\varphi_{1,2}$ represented on the figure, and three complementary "steering" wheel angular velocities which provide extra degrees of freedom. In Morin and Samson (2010), we have applied the transverse function control approach to the more difficult case when only the articulation angular velocities can be changed. This case had previously been considered in Ishikawa (2009), in the open-loop control context to illustrate the possibility of switching between a set of piecewise sinusoidal inputs to produce a desired net displacement effect. We here address the intermediary case of two steering wheels. Beyond the practical advantage resulting from two extra control inputs which, even though they are not needed for the system's controllability, allow for smoother displacements with less maneuvers, this case gives us the opportunity to illustrate the use of transverse functions defined on $\mathbb{SO}(4)$ on a practical example.

The paper is organized as follows. Notation and a recall of the Transverse Function Theorem are provided in Section 2. The robot's kinematic model and control objectives are presented in Section 3. The main results are stated in Section 4 where a control design methodology based on the application of the TF approach is proposed. A control error model is first specified. Then, a controllable homogeneous approximation of this model and an associated transverse function defined on $\mathbb{SO}(4)$ are derived. Finally, a feedback control solution using this function, with proved stabilization properties, is presented. The validity and performance of the proposed controller are demonstrated in Section 5 with illustrative simulation results. Section 6 points out a few research directions which could extend the present study.

2. NOTATION AND RECALL OF THE TRANSVERSE FUNCTION THEOREM

In this paper, x' denotes the transpose of a vector $x \in \mathbb{R}^n$, and $|x|$ its euclidean norm. I_n is the identity matrix of dimension $(n \times n)$, and $O_{m\times n}$ is the zero-valued matrix of dimension $(m \times n)$. The *ith* component of the vector x is denoted as x_i. It is assumed that the reader is familiarized with basic properties of systems on Lie groups. We refer him, e.g., to Morin and Samson (2009a) for more details in the context of the control of nonholonomic systems.

Notions about Transverse Functions are now recalled –see e.g. Morin and Samson (2003) for more details. Let $\mathbf{X} = \{X_1, \dots, X_m\}$ denote a family of smooth v.f. $X_1, \dots, X_m$ on a n-dimensional manifold M whose tangent space at the point $x \in M$ is denoted as T_xM . $\mathbf{X}$ satisfies the Lie Algebra Rank Condition (LARC) at some point q_0 if $\mathrm{Lie}(\mathbf{X})(q_0) = T_{q_0}M$ with

$$\mathrm{Lie}(\mathbf{X}) = \mathrm{span}\{X_i, [X_i, X_j], [X_i, [X_j, X_k] \dots \quad i, j, k, \dots = 1, \dots, m\}$$

and $\mathrm{Lie}(\mathbf{X})(q) = \{X(q) : X \in \mathrm{Lie}(\mathbf{X})\}$. Given a compact manifold K, a smooth function $f : K \longrightarrow M$ is *transverse to* $\mathbf{X}$ if, for any $\alpha \in K$,

$$\mathrm{span}\{X_1(f(\alpha)), \dots, X_m(f(\alpha)), df(\alpha)(T_\alpha K)\} = T_{f(\alpha)}M \qquad (1)$$

with df the differential of f. Note that the dimension of K must be at least equal to $(n-m)$. Given $q_0 \in M$ such that the family $\mathbf{X}$ satisfies the LARC at q_0, the "Transverse Function theorem" in Morin and Samson (2001) ensures the existence of a family $(f^\varepsilon)_{\varepsilon>0}$ of functions transverse to $\mathbf{X}$, with $\max_\alpha \mathrm{dist}(f^\varepsilon(\alpha), q_0) \to 0$ as $\varepsilon \to 0$, where "dist" denotes any distance locally defined in the neighborhood of q_0.

3. MODELING AND CONTROL OBJECTIVES

The wheeled snake mechanism under consideration is depicted on Fig. 1. It is composed of three wheeled "segments" connected by two actuated rotoid articulations, and it differs from the mechanism studied in Morin and Samson (2010) by the added steering-wheel rotoid articulations allowing for the modification of the angles γ_1 and γ_2. From a mechanical point of view, this system is alike a unicycle vehicle located between two trailers with off-axle trailer hitches and complementary steering wheels. However, alike the case where γ_1 and γ_2 are kept equal to zero, and due to the existence of mechanical singularities, actuating the joint angles $\varphi_{1,2}$ is not strictly equivalent to actuating the longitudinal and angular velocities of one of the vehicles. This actuation particularity makes an important difference at the control level, and it underlies the serpentine locomotion mode which allows the system to be displaced without encountering mechanical singularities (the characterization of which is addressed a little further).

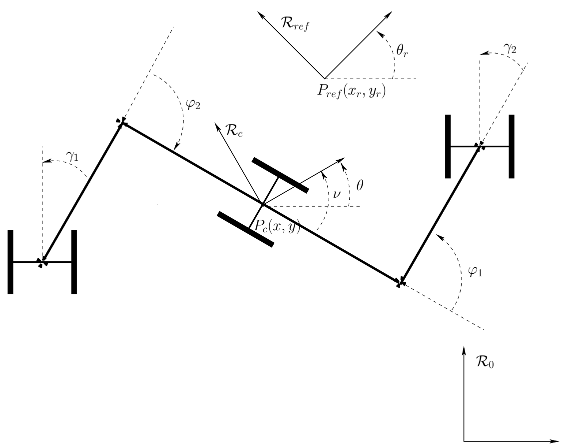

Fig. 1. Three-segments snake robot with two steering wheels

Given an inertial frame $\mathcal{R}_0$ and a body-fixed frame $\mathcal{R}_c$, here attached for symmetry reasons to the snake's mid-segment, the configuration of this segment in cartesian space is given by

$$g = \begin{pmatrix} p \\ \theta \end{pmatrix} \in \mathbb{SE}(2) \quad , \; p = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \quad , \; \theta \in \mathbb{T}$$

with x and y the coordinates of the point P_c (the origin of $\mathcal{R}_c$ located on the wheels' axle at mid-distance of the rotoid articulations) in $\mathcal{R}_0$, and θ the orientation of $\mathcal{R}_c$ w.r.t. $\mathcal{R}_0$. The orientation angle of $\mathcal{R}_c$ w.r.t. this segment is denoted as ν. As already pointed out in Morin and Samson (2010), the choice of this angle is related to the fact that some values are better than others in terms of control and singularity avoidance.

The “shape” of the snake robot depends on the shape angle vector

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{T}^2$$

and the steering-wheel angle vector

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathbb{T}^2$$

The complete configuration vector of the system is thus given by $(g, \varphi, \gamma) \in \mathbb{SE}(2) \times \mathbb{T}^2 \times \mathbb{T}^2$. Define the following change of coordinates

$$\varphi \longmapsto \bar{\varphi} = \begin{pmatrix} \tan \dfrac{\varphi_1}{2} \\ \tan \dfrac{\varphi_2}{2} \end{pmatrix}, \quad \gamma \longmapsto \bar{\gamma} = \begin{pmatrix} \tan \gamma_1 \\ \tan \gamma_2 \end{pmatrix} \tag{2}$$

Under the classical non-slipping assumption associated with wheel-ground contact, and assuming for the sake of notation simplicity that the distances between the three wheels'axles and adjacent rotoid articulations are all equal to one, it is not difficult to extend the kinematic model derived in Morin and Samson (2010) to the case where γ_1 and γ_2 are different from zero. This yields the following driftless control model

$$\begin{cases} \dot{g} &= B(\theta - \nu)v \\ \dot{\bar{\varphi}} &= A(\bar{\varphi}, \bar{\gamma})v \\ \dot{\bar{\gamma}} &= u_\gamma \end{cases} \tag{3}$$

with

$$B(\alpha) = \begin{pmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & 1 \end{pmatrix}$$

$v = (v_l, \dot{\theta})'$ the vector regrouping the linear and angular velocities of the mid-segment, $u_\gamma = (\dot{\gamma}_1 / \cos(\gamma_1)^2, \dot{\gamma}_2 / \cos(\gamma_2)^2)'$, and

$$A(\bar{\varphi}, \bar{\gamma}) = \begin{pmatrix} \bar{\varphi}_1 + \dfrac{1 - \bar{\varphi}_1^2}{2} \bar{\gamma}_1 & -1 + \bar{\varphi}_1 \bar{\gamma}_1 \\ -\bar{\varphi}_2 + \dfrac{1 - \bar{\varphi}_2^2}{2} \bar{\gamma}_2 & 1 + \bar{\varphi}_2 \bar{\gamma}_2 \end{pmatrix} \tag{4}$$

One can verify that the matrix A is singular when the three wheels' axles intersect at the same point. The corresponding shape and steering-wheel angles, complemented with either γ_1 or γ_2 equal to $\pm\frac{\pi}{2}$, and shape angles $\varphi_1 = \varphi_2 = \pi$, form the set of the mechanical singularities of the system. The necessity of not crossing this set justifies in part the choice of the above change of angle coordinates. One also verifies that, among the infinite number of configurations (φ, γ) for which this matrix is invertible, a subset is composed of shape angles $\varphi_1 \neq \varphi_2$ $(\neq \pi)$ and zero steering-wheel angles, i.e. $\gamma_1 = \gamma_2 = 0$; a fact already used in Morin and Samson (2010). Therefore, provided that the shape and steering-wheel angles remain close to this subset, the matrix A remains invertible, and mechanical singularities are avoided. This observation suggests to choose a *reference* or *nominal* vector (φ_r, γ_r) for the shape and steering-wheel angles for which the matrix A is well-defined and invertible, and determine a control policy which i) maintains these angles near their nominal values and ii) stabilizes in a ”practical sense” any given reference trajectory $g_r(t) = (x_r(t), y_r(t), \theta_r(t))'$ for the mechanism's posture g. Here, the notion of ”practical” stabilization should not be interpreted as a relaxed control objective, but rather as a requirement resulting from the necessity to avoid mechanical singularities. Consider, for instance, the case where $\nu = 0$ and the reference trajectory is a straight line with $\dot{x}_r(t)$ and $\dot{y}_r(t)$ constant and not both equal to zero. Perfect (asymptotic) tracking then implies that all segments are aligned with zero steering-wheel angles: a configuration which is singular for this mechanism as a consequence of its specific means of actuation. Therefore, it is compulsory to accept a bounded (and eventually small) tracking error so as to render singularity avoidance possible.

4. CONTROL DESIGN

4.1 Error system

We now determine an *error system* whose zero-state ”practical” stabilization achieves the control objective discussed previously. Since g and $g_r(t)$ are elements of the Lie group $\mathbb{SE}(2)$, with the group product defined by

$$g_1 g_2 = \begin{pmatrix} p_1 + Q(\theta_1) p_2 \\ \theta_1 + \theta_2 \end{pmatrix}$$

with $Q(\theta)$ the rotation matrix in the plane of angle θ, it is natural (and recommended) to characterize the ”error” between these elements by another element of the group. A possible candidate is

$$\tilde{g} = \begin{pmatrix} p_r \\ \theta_r - \nu \end{pmatrix}^{-1} \begin{pmatrix} p \\ \theta - \nu \end{pmatrix} = \begin{pmatrix} Q(\nu - \theta_r)(p - p_r) \\ \theta - \theta_r \end{pmatrix}$$

Denoting the nominal shape angle vectors as $\varphi_r = (\varphi_{r,1}, \varphi_{r,2})'$ and $\bar{\varphi}_r = (\tan\frac{\varphi_{r,1}}{2}, \tan\frac{\varphi_{r,2}}{2})'$, setting $\tilde{\bar{\varphi}} = \bar{\varphi} - \bar{\varphi}_r$, and taking $\gamma_r = (0,0)'$ as the nominal values for the steering-wheel angles, one obtains the following *error system*

$$\begin{cases} \dot{\tilde{g}} &= B(\tilde{g}_3)v + b_g(\tilde{g}, \dot{g}_r) \\ \dot{\tilde{\bar{\varphi}}} &= A(\bar{\varphi}_r + \tilde{\bar{\varphi}}, \bar{\gamma})v - \dot{\bar{\varphi}}_r \\ \dot{\bar{\gamma}} &= u_\gamma \end{cases} \tag{5}$$

with

$$b_g(\tilde{g}, \dot{g}_r) = -\begin{pmatrix} Q(\nu - \theta_r) & \begin{pmatrix} -\tilde{g}_2 \\ \tilde{g}_1 \end{pmatrix} \\ 0_{1\times 2} & 1 \end{pmatrix} \begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta}_r \end{pmatrix}$$

the additive “perturbation” arising from the motion of the reference frame. This is a perturbed driftless system with four control inputs –the components of v and u_γ. One easily verifies that, at the origin ($\tilde{g} = 0_{3\times1}, \tilde{\bar{\varphi}} = 0_{2\times1}, \bar{\gamma} = 0_{2\times1}$), the distribution generated by the four control vector fields (v.f.) and first-order Lie brackets of these v.f. is a seven-dimensional vector space. Therefore, the local controllability at the origin of the error-system is obtained with Lie brackets of order up to one only.

Remark: The error system is locally controllable even at points where both shape angles φ_1 and φ_2 are equal to zero, and the steering-wheel angles γ_1 and γ_2 are the opposite of each other, i.e. at points where the matrix A is not invertible. This means that if v could be controlled directly, then one could work out feedback (practical) stabilizers around zero shape angles and, for instance, zero steering-wheel angles. The necessity of keeping A invertible all the time is a consequence of the specific actuation considered here.

From Morin and Samson (2009b), the above-mentioned structural properties of the system implies the existence of transverse functions for this system that are defined on the special orthogonal group $\mathbb{SO}(4)$. These functions present the advantage over other functions defined of the 3-dimensional torus of respecting the structural symmetries of the system's Control Lie Algebra. On the other hand, the "price" paid for this advantage is the larger dimension –equal to six in the present case– of the manifold on which the transverse function is defined or, in other words, the larger dimension of the dynamic extension used to solve the control problem at hand. There are various ways to characterize such a function. The first possibility consists in considering the one defined by

$$f^{\varepsilon}(R) = \exp(\varepsilon \sum_{i=1}^{4} \alpha_i X_i + \frac{\varepsilon^2}{2} \sum_{1 \leq i < j \leq 4} \beta_{i,j} [X_i, X_j]) \tag{6}$$

with
$\alpha = Re_1$
$\beta = c_1(Re_1) \wedge (Re_2) + c_2(Re_2) \wedge (Re_3)$
where:

- $X_{1,2,3,4}$ are the control v.f. of the system,
- $R \in \mathbb{SO}(4)$, $\varepsilon > 0$, c_1 and c_2 are non-zero real numbers,
- $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of $\mathbb{R}^4$,
- $\wedge$ denotes the wedge product associated with $\mathbb{R}^4$,
- $\beta_{i,j}$ is the (i, j) component of β decomposed in the canonical basis $\{e_i \wedge e_j\}$ $(1 \leq i < j \leq 4)$ of the six-dimensional vector space $\bigwedge^2 \mathbb{R}^4$, i.e. $\beta = \sum_{1 \leq i < j \leq 4} \beta_{i,j} e_i \wedge e_j$,
- $\exp(X)$ is the solution at time $t = 1$ to the equation $\dot{x} = X(x)$ with $x(0) = 0$.

This function is transverse at the system's origin in the sense of the definition recalled in Section 2 provided that ε is small enough. A problem with the expression (6) is that, in the present case, the exponential operation involves an integration which does not yield an explicit expression in terms of classical functions. A systematic way to circumvent this problem (see Morin and Samson (2003) for more detailed explanations) consists in first considering a locally controllable (nilpotent) homogeneous approximation (Stefani (1985), Hermes (1991)) of the unperturbed part of the error system, and calculate the above exponential for this homogeneous system. The expression of this function can be explicit because the integration process involves polynomial functions exclusively. From there, it suffices to transform this function via the inverse of the change coordinates involved in the characterization of the homogeneous approximation to obtain a transverse function for the original system.

This procedure is next carried out in more details, prior to addressing the control design itself.

4.2 Homogeneous approximation and transverse functions on $\mathbb{SO}(4)$

Consider the change of coordinates

$$\Psi \; : \; \begin{pmatrix} \tilde{g} \\ \tilde{\bar{\varphi}} \\ \bar{\gamma} \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{g} \\ \eta \\ \bar{\gamma} \end{pmatrix}, \quad \text{with}$$
$$\eta = \tilde{\bar{\varphi}} + \begin{pmatrix} -\bar{\varphi}_{r,1} \\ \bar{\varphi}_{r,2} \end{pmatrix} \tilde{g}_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tilde{g}_3 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tilde{g}_2 - 0.5 \bar{\varphi}_r \tilde{g}_1^2 \tag{7}$$

One verifies that an homogeneous approximation of degree zero, with weight vector $r = (1, 2, 1, 2, 2, 1, 1)$, of the unperturbed part of system (5) –i.e. when b_g and $\dot{\bar{\varphi}}_r$ are equal to zero– transformed by this change of coordinates is

$$\begin{cases} \dot{x}_{1,2,3} = \begin{pmatrix} 1 & 0 \\ x_3 & 0 \\ 0 & 1 \end{pmatrix} v \\ \dot{x}_{4,5} \;\; = \begin{pmatrix} \frac{1 - \bar{\varphi}_{r,1}^2}{2} x_6 & \bar{\varphi}_{r,1} x_6 \\ \frac{1 - \bar{\varphi}_{r,2}^2}{2} x_7 & \bar{\varphi}_{r,2} x_7 \end{pmatrix} v \\ \dot{x}_{6,7} \;\; = u_{\gamma} \end{cases} \tag{8}$$

This system may also be written as

$$\begin{aligned} &\dot{x} = X_1(x) v_1 + X_2(x) v_2 + X_3 u_{\gamma,1} + X_4 u_{\gamma,2} \\ &\text{with} \\ &X_1(x) = (1, x_3, 0, \frac{1 - \bar{\varphi}_{r,1}^2}{2} x_6, \frac{1 - \bar{\varphi}_{r,2}^2}{2} x_7, 0, 0)' \\ &X_2(x) = (0, 0, 1, \bar{\varphi}_{r,1} x_6, \bar{\varphi}_{r,2} x_7, 0, 0)' \\ &X_3 = (0, 0, 0, 0, 0, 1, 0)' \\ &X_4 = (0, 0, 0, 0, 0, 0, 1)' \end{aligned} \tag{9}$$

One also verifies that the Control Lie Algebra of this system is generated by the four control v.f. $X_{1,2,3,4}$, the Lie braket $X_5 = [X_1, X_2]$, and two other first-order Lie brackets among $X_6 = [X_1, X_3]$, $X_7 = [X_1, X_4]$, $X_8 = [X_2, X_3]$, $X_9 = [X_2, X_4]$. It is thus a seven-dimensional vector space on $\mathbb{R}$. This in turn implies that the homogeneous system is left-invariant on $\mathbb{R}^7$ with respect to some group operation. In fact, it is not difficult to verify that this group operation is given by

$$x \star y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 + x_3 y_1 \\ x_3 + y_3 \\ x_4 + y_4 + (\frac{1 - \bar{\varphi}_{r,1}^2}{2} y_1 + \bar{\varphi}_{r,1} y_3) x_6 \\ x_5 + y_5 + (\frac{1 - \bar{\varphi}_{r,2}^2}{2} y_1 + \bar{\varphi}_{r,2} y_3) x_7 \\ x_6 + y_6 \\ x_7 + y_7 \end{pmatrix} \tag{10}$$

The conditions specified in Morin and Samson (2009b), according to which it is possible to derive for this system a transverse function on $\mathbb{SO}(4)$ –the number four corresponding to the number of independent control inputs– are thus met. Moreover, such a function is given by (6) with X_i $(i = 1, 2, 3, 4)$ denoting now the v.f. of the homogeneous approximation. One can further verify from the proof of transversality given in Morin and Samson (2009b) that, in the particular case where the Lie bracket $[X_3, X_4]$ is null –as in the present case–, the number c_2 involved in (6) can be taken equal to zero. By using the expressions (9) of the system's v.f., the explicit calculation of this function yields

$$h^\varepsilon(R) = \begin{pmatrix} h^\varepsilon_g(R) \\ h^\varepsilon_\eta(R) \\ h^\varepsilon_\gamma(R) \end{pmatrix}, \quad \text{with}$$

$$h^\varepsilon_g = \begin{pmatrix} \varepsilon\alpha_1 \\ \frac{\varepsilon^2}{2}(\alpha_1\alpha_2 - \beta_{1,2}) \\ \varepsilon\alpha_2 \end{pmatrix}$$

$$h^\varepsilon_\eta = \begin{pmatrix} \frac{\varepsilon^2}{2}\left(\frac{1-\bar\varphi^2_{r,1}}{2}(\alpha_1\alpha_3 - \beta_{1,3}) + \bar\varphi_{r,1}(\alpha_2\alpha_3 - \beta_{2,3})\right) \\ \frac{\varepsilon^2}{2}\left(\frac{1-\bar\varphi^2_{r,2}}{2}(\alpha_1\alpha_4 - \beta_{1,4}) + \bar\varphi_{r,2}(\alpha_2\alpha_4 - \beta_{2,4})\right) \end{pmatrix}$$

$$h^\varepsilon_\gamma = \begin{pmatrix} \varepsilon\alpha_3 \\ \varepsilon\alpha_4 \end{pmatrix} \tag{11}$$

A transverse function for the error-system (5) is then

$$f^\varepsilon(R) = \begin{pmatrix} f^\varepsilon_g(R) \\ f^\varepsilon_\varphi(R) \\ f^\varepsilon_\gamma(R) \end{pmatrix} = \Psi^{-1}(h^\varepsilon(R)), \quad \text{with}$$

$$\begin{aligned} f^\varepsilon_g &= h^\varepsilon_g \\ f^\varepsilon_\varphi &= h^\varepsilon_\eta - \begin{pmatrix} -\bar\varphi_{r,1} \\ \bar\varphi_{r,2} \end{pmatrix} h^\varepsilon_{g,1} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} h^\varepsilon_{g,3} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} h^\varepsilon_{g,2} \\ &\quad + 0.5\bar\varphi_r {h^\varepsilon_{g,1}}^2 \\ f^\varepsilon_\gamma &= h^\varepsilon_\gamma \end{aligned} \tag{12}$$

and ε chosen "small enough" (but different from zero). To simplify, we assume from now on that the nominal shape angle vector φ_r is constant, so that $\dot\varphi_r = \dot{\bar\varphi}_r = 0$. The extension of the results to the case when this assumption is not made poses no theoretical difficulty. Let $\omega \in \mathbb{R}^6$ denote the instantaneous angular velocity vector associated with the time-derivative of R, i.e. the vector such that

$$\frac{d}{dt}R = R\sum_{i=1}^{6} S_i\omega_i$$

with $\{S_i\}_{i=1,\dots,6}$ denoting a set of independent (4×4) elementary skew-symmetric matrices which constitutes a basis of the tangent space of $\mathbb{SO}(4)$ at the identity. Let us denote as $d_R f^\varepsilon$ the differential of f^ε such that

$$\frac{d}{dt}f^\varepsilon(R) = d_R f^\varepsilon(R)\omega$$

The calculation of this differential from (11), (12), and the definitions of α and β, poses no difficulty. The property of transversality means that the (7×10) matrix

$$\begin{pmatrix} B(f^\varepsilon_{g,3}) & O_{3\times 2} & d_R f^\varepsilon_g \\ A(\bar\varphi_r + f^\varepsilon_\varphi, f^\varepsilon_\gamma) & O_{2\times 2} & d_R f^\varepsilon_\varphi \\ O_{2\times 2} & I_2 & d_R f^\varepsilon_\gamma \end{pmatrix}(R)$$

is of full rank (equal to seven) $\forall R \in \mathbb{SO}(4)$. Since, by choice of $\bar\varphi_r$, the matrix $A(\bar\varphi_r, 0)$ is invertible, and since $f^{\varepsilon=0} = 0$ by construction of the transverse function, there exists $\varepsilon_0 > 0$ such that $A(\bar\varphi_r + f^\varepsilon_\varphi, f^\varepsilon_\gamma)(R)$ is invertible $\forall R \in \mathbb{SO}(4)$, when $0 \le \varepsilon < \varepsilon_0$. The transversality property then implies that the (3×6) matrix

$$C(R) = (B(f^\varepsilon_{g,3})A(\bar\varphi_r + f^\varepsilon_\varphi, f^\varepsilon_\gamma)^{-1} d_R f^\varepsilon_\varphi - d_R f^\varepsilon_g)(R) \tag{13}$$

is of full rank (equal to three) $\forall R \in \mathbb{SO}(4)$, when $0 < \varepsilon < \varepsilon_0$. This latter property and the fact that $A(\bar\varphi_r + f^\varepsilon_\varphi, f^\varepsilon_\gamma)$ is invertible when ε is small enough are central to the feedback control design proposed next.

4.3 Transverse function control design

Define

$$z = \begin{pmatrix} z_g \\ z_\varphi \\ z_\gamma \end{pmatrix} = \begin{pmatrix} \tilde g (f^\varepsilon_g)^{-1} \\ \tilde{\bar\varphi} - f^\varepsilon_\varphi \\ \bar\gamma - f^\varepsilon_\gamma \end{pmatrix} \tag{14}$$

The objective is now to determine a feedback control which asymptotically stabilizes $z = 0$, knowing that the satisfaction of this objective implies that the tracking error $\tilde g$ will ultimately remain small (practical stabilization), whereas φ and γ will stay close to their respective nominal values φ_r and $(0,0)'$, provided that ε is chosen small. This will in turn ensure the avoidance of mechanical singularities.
In view of (5)

$$\dot z_\gamma = u_\gamma - d_R f^\varepsilon_\gamma \omega$$

This relation suggests to set, for instance

$$u_\gamma = d_R f^\varepsilon_\gamma \omega - k_\gamma z_\gamma \tag{15}$$

with $k_\gamma > 0$ the control gain which determines the closed-loop exponential rate of convergence of z_γ to zero. Moreover, if the initial values of $|\gamma_1|$ and $|\gamma_2|$ are small, then this control ensures that these angles remain small thereafter, when ε is itself chosen small.
Let us now consider the stabilization of z_φ. In view of (5), and since $\dot{\bar\varphi}_r = 0$ by assumption

$$\dot z_\varphi = A(\bar\varphi, \bar\gamma)v - d_R f^\varepsilon_\varphi \omega$$

This relation suggests to set, for instance

$$v = A(\bar\varphi, \bar\gamma)^{-1}(d_R f^\varepsilon_\varphi \omega - k_\varphi z_\varphi) \tag{16}$$

with $k_\varphi > 0$ the control gain which determines the closed-loop exponential rate of convergence of z_φ to zero. Let us finally consider the evolution of z_g. Denoting by l_g and r_g the left and right translations by g on the Lie group $\mathbb{SE}(2)$, i.e. the operations defined by $l_{g_1}(g_2) = r_{g_2}(g_1) = g_1 g_2$, and by dl_g and dr_g the corresponding differential given by (see Morin and Samson (2009a), for instance)

$$dl_{g_1}(g_2) = \begin{pmatrix} Q(\theta_1) & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{pmatrix} \quad (= \bar Q(\theta_1))$$

and

$$dr_{g_2}(g_1) = \begin{pmatrix} I_2 & Q(\theta_1)\begin{pmatrix} -y_2 \\ x_2 \end{pmatrix} \\ 0_{1\times 2} & 1 \end{pmatrix}$$

one has, using the left-invariance of $B(.)$ with respect to the group operation on $\mathbb{SE}(2)$

$$\begin{aligned} \dot z_g &= dr_{f_g^{\varepsilon\,-1}}(\tilde g)(\dot{\tilde g} - dl_{z_g}(f^\varepsilon_g)\dot f^\varepsilon_g) \\ &= dr_{f_g^{\varepsilon\,-1}}(\tilde g) dl_{z_g}(f^\varepsilon_g)(B(f^\varepsilon_{g,3})v - d_R f^\varepsilon_g \omega) + dr_{f_g^{\varepsilon\,-1}}(\tilde g) b_g \end{aligned}$$

Therefore, on the zero dynamics ($z_\gamma = 0, z_\varphi = 0$) (to which the system's solutions converge when applying the feedback controls u_γ and v defined by (15) and (16)) respectively)

$$v = A(\bar\varphi_r + f^\varepsilon_\varphi, f^\varepsilon_\gamma)^{-1} d_R f^\varepsilon_\varphi \omega$$

and

$$\dot z_g = \bar C\omega + dr_{f_g^{\varepsilon\,-1}}(\tilde g) b_g$$

with

$$\bar C = dr_{f_g^{\varepsilon\,-1}}(\tilde g) dl_{z_g}(f^\varepsilon_g) C$$

and C the matrix defined in (13). Since the differentials dr and dl are invertible by definition, and since the matrix C is of full rank (equal to three) as a consequence of the transversality property of the function f^ε when ε is small enough, the matrix $\bar{C}$ is itself of full-rank when ε is small enough. The last but one previous relation then suggests to set, for instance

$$\omega = -\bar{C}^{\dagger}(dr_{f_g^{\varepsilon}-1}(\tilde{g})b_g + k_g z_g) \tag{17}$$

with $\bar{C}^{\dagger}$ a right pseudo-inverse of $\bar{C}$ and $k_g > 0$ the control gain which determines the closed-loop exponential rate of convergence of z_g to zero.
The following proposition summarizes the control design and its stabilizing properties

Proposition 1. Assume that the reference velocity $\dot{g}_r$ is bounded and that the nominal shape angle vector φ_r does not correspond to a mechanical singularity when the steering-wheel angles $\gamma_{1,2}$ are equal to zero, i.e. is such that $A(\varphi_r, 0)$ is invertible. Then there exist two positive numbers ε_1 and ε_2 such that i) the function f^{ε} on $\mathbb{SO}(4)$ given by (12) is transverse to the v.f. of the error system (5) and ii) the feedback control given by (15)-(17) exponentially stabilizes $z = 0$ when $|\varphi(0) - \varphi_r| < \varepsilon_1$ and $0 < \varepsilon < \varepsilon_2$.

Remarks:

- the asymptotic stability of $z = 0$ is global with respect to position/orientation error between the chosen frame attached to the mechanism's central link and the reference frame.
- The proposition applies with no condition upon the choice of the "stance" angle ν which influences the way the mechanism moves when tracking the reference frame. In Morin and Samson (2010), the choice of this angle could be related to the curvature of the reference trajectory in order to limit motion displacements along directions specified by second-order Lie brackets and, in doing so, to avoid complicated maneuvers when possible. In the present case, second-order Lie brackets are not needed to the system's controllability so that the guidance rule proposed in the case where the steering-wheel angles are fixed does not apply. It thus remains to find out a criterion which could be used to calculate an "optimal" value for this angle. In particular, the possibility of relating this issue to control energy aspects has not yet been explored.

5. SIMULATION RESULTS

For these simulations we have set $\nu = \frac{\pi}{3}$, $\varphi_r = (\frac{2\pi}{3}, -\frac{2\pi}{3})'$, $\gamma_r = (0, 0)'$, and used the feedback control (15)-(17) with gains $k_\varphi = k_\gamma = 3$ and $k_g = 1$. The transverse function used in the control law was calculated according to (11) and (12) with $\varepsilon = 0.6$. The coefficients involved in the expression of β given in (6) are $c_1 = 1$ and $c_2 = 0$. A more precise tracking could be achieved by choosing a smaller value for ε, but this would yield higher-frequency maneuvers (body deformations) and involve larger velocity inputs. For the reported simulation, the time history of the reference frame velocity $v_r = \bar{Q}(-\theta_r)\dot{g}_r$ is summarized in the following table.

$t \in (s)$	$v_r =$ (m/s, m/s, rad/s)$'$
$[0, 5)$	$(0, 0, 0)'$
$[5, 13)$	$(0.6, 0, 0.05(t-5))'$
$[13, 21)$	$(0.6, 0, 4 - 0.05(t-13))'$
$[21, 28)$	$(0.8, 0, 0)'$
$[28, 37)$	$(0, 0, 0.4)'$
$[37, 45)$	$(0, -0.5, 0)'$
$[45, 51)$	$(-1.5, 0, 0.6)'$
$[51, 55)$	$(-1.5, 0, -0.6)'$
$[55, 60)$	$(0, 0, 0)'$

For comparison purposes, it is the same as the one used in Morin and Samson (2010) in the case where the wheel angles $\gamma_{1,2}$ are constant equal to zero. The exponential convergence to zero of the (euclidean) norm of z can be observed on Fig. 2. The (x, y) trajectories of the origin

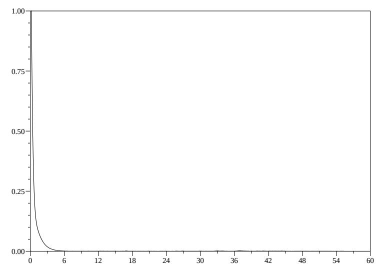

Fig. 2. $\|z\|$ vs. time

of the reference frame (dotted line) and of the origin P_c of the robot's frame (dashed line) are shown on Fig. 3, with superposed snapshots taken every ten seconds of the wheeled mechanism and of the reference frame that it is tracking. The principle of practical stabilization and

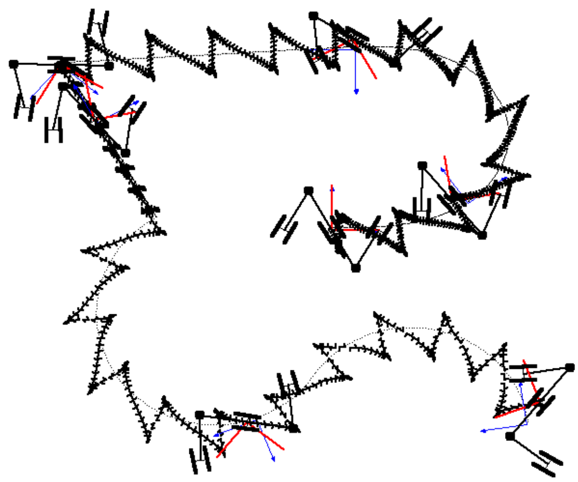

Fig. 3. Reference trajectory $(x_r(t), y_r(t))$ and snake's trajectory $(x(t), y(t))$

tracking is well illustrated by this figure. However, only a video of the simulation can qualitatively report of the "natural" character of the mechanism's deformations in all motion phases.

The time-evolution of the shape angles $\varphi_{1,2}$ about their nominal values is shown on Fig. 4 and the time-evolution

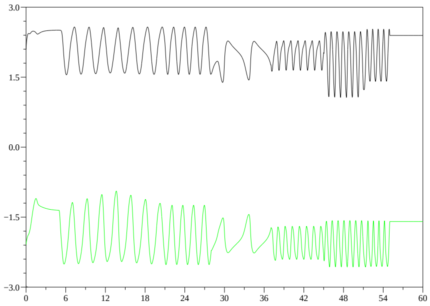

Fig. 4. Shape angles $\varphi_{1,2}$ vs. time

of the steering-wheel angles about their zero nominal value is shown on Fig. 5 The quasi-periodicity and continuous

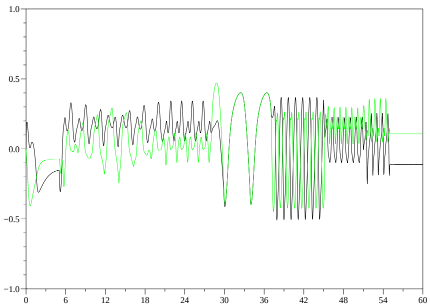

Fig. 5. Steering-wheel angles $\gamma_{1,2}$ vs. time

adaptation of the variations of the shape and steering-wheel angles are noticeable. One also verifies that no mechanical singularity is crossed by observing that the determinant of the matrix $A(\bar{\varphi}, \bar{\gamma})$ is always strictly positive, as shown on Fig. 6.

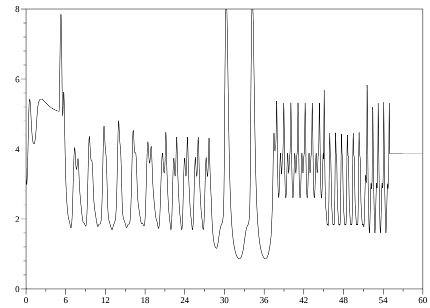

Fig. 6. $det(A(\bar{\varphi}, \bar{\gamma}))$ vs. time

6. CONCLUDING REMARKS

The control of a nonholonomic snake-like wheeled mechanism has been addressed with the objective of achieving the practical stabilization of any reference trajectory in cartesian space, while maintaining the mechanism's internal variables –which characterize the shape of the mechanism– away from mechanical singularities. The control design relies on the Transverse Function approach and exploits actuation redundancy to reduce control input intensities. To this purpose transverse functions defined on the special orthogonal group $\mathbb{SO}(4)$ have been used and applied for the first time –to our knowledge– to the control of a physical system. Future work includes, e.g., the study of snake-like mechanisms with more links, alike Hirose's ACM III snake robot, and the control of mechanisms with both kinematic (nonholonomic) and dynamic constraints.

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APPENDIX: REPRESENTATION OF ELEMENTS OF $\mathbb{SO}(4)$ BY PAIRS OF QUATERNIONS AND ASSOCIATED DIFFERENTIAL RELATIONS

A classical theorem, Ball (1889), Bouman (1932), Mebius (1994), states that every rotation matrix in $\mathbb{SO}(4)$ is the product of two *isoclinic* rotations, i.e

$$R \in \mathbb{SO}(4) \;\Rightarrow R = R_l R_r,\; (R_l, R_r) \in \mathbb{SO}(4) \times \mathbb{SO}(4)$$

with

$$R_l = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \quad , \; a^2 + b^2 + c^2 + d^2 = 1$$

a *left-isoclinic* rotation, and

$$R_r = \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix} \quad , \; p^2 + q^2 + r^2 + s^2 = 1$$

a *right-isoclinic* rotation. Moreover this decomposition is unique up to the sign multiplying the isoclinic rotation matrices. Therefore every rotation in $\mathbb{SO}(4)$ can be represented by the pair of unitary quaternions (q_l, q_r) given by

$$q_l = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} q_{l,0} \\ \bar{q}_l \end{pmatrix} \text{ with } q_{l,0} \in \mathbb{R},\; \bar{q}_l \in \mathbb{R}^3$$

and

$$q_r = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} q_{r,0} \\ \bar{q}_r \end{pmatrix} \text{ with } q_{r,0} \in \mathbb{R},\; \bar{q}_r \in \mathbb{R}^3$$

One verifies by a direct calculation that $\forall x \in \mathbb{R}^4$

$$Rx = q_l.x.q_r \tag{18}$$

where $(.)$ denotes the product of quaternions, i.e. the operation defined by

$$\begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} . \begin{pmatrix} y_0 \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x_0 y_0 - \bar{x}'\bar{y} \\ \bar{x} \times \bar{y} + x_0 \bar{y} + y_0 \bar{x} \end{pmatrix}$$

with $(\times)$ denoting the cross-product in $\mathbb{R}^3$. Recall that i) the instantaneous velocity vector $\bar{\omega}$ associated with a rotation $Q \in \mathbb{SO}(3)$ is such that $\dot{Q} = QS(\bar{\omega})$ with $S(.)$ the matrix-valued function associated with the cross-product, i.e. such that $x \times y = S(x)y$, and ii) by denoting the quaternion associated with Q as q one has $\dot{q} = \frac{1}{2} q.\omega$ with $\omega = (0, \bar{\omega}')'$ a pure imaginary quaternion.

Lemma 2. Let ω denote a 6-dimensional instantaneous velocity vector associated with the variations of a rotation $R \in \mathbb{SO}(4)$, i.e. $\dot{R} = R\sum_{i=1}^{6} S_i \omega_i$ with $\{S_i\}$ denoting a basis of (4×4) unitary skew-symmetric matrices. Let (q_l, q_r) denote the pair of unitary quaternions used to represent R, and ω_l and ω_r the pure imaginary quaternions associated with the variations of q_l and q_r respectively, i.e. $\dot{q}_l = \frac{1}{2} q_l.\omega_l$ and $\dot{q}_r = \frac{1}{2} q_r.\omega_r$ with $\omega_l = (0, \bar{\omega}_l')'$ and $\omega_r = (0, \bar{\omega}_r')'$. Then

$$\forall x \in \mathbb{R}^4 \;:\; (\sum_{i=1}^{6} S_i \omega_i)x = \frac{1}{2}(\omega_l.x + x.q_r.\omega_r.q_r^{-1}) \tag{19}$$

By defining $\bar{\omega}$ and $\bar{\bar{\omega}}$ as the vectors in $\mathbb{R}^3$ such that

$$\sum_{i=1}^{6} S_i \omega_i = \begin{pmatrix} 0 & -\bar{\omega}' \\ \bar{\omega} & S(\bar{\bar{\omega}}) \end{pmatrix}$$

and by denoting as $Q(q_r)$ the rotation matrix in $\mathbb{SO}(3)$ whose associated quaternion is q_r, the above relation in turn implies that

$$\begin{aligned} \bar{\omega}_l &= \bar{\omega} + \bar{\bar{\omega}} \\ \bar{\omega}_r &= Q(q_r)'(\bar{\omega} - \bar{\bar{\omega}}) \end{aligned} \tag{20}$$

Proof:
Part i): Relation (19) is obtained by differentiating both members of (18) with respect to time

$$\begin{aligned} \frac{d}{dt}(Rx) &= R(\sum_{i=1}^{6} S_i \omega_i)x \\ &= q_l.(\sum_{i=1}^{6} S_i \omega_i)x.q_r \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(q_l.x.q_r) &= \dot{q}_l.x.q_r + q_l.x.\dot{q}_r \\ &= \frac{1}{2}(q_l.\omega_l.x.q_r + q_l.x.q_r.\omega_r) \\ &= q_l\big(\frac{1}{2}(\omega_l.x + x.q_r.\omega_r.q_r^{-1})\big)q_r \end{aligned}$$

Part ii): By using the known relation

$$q.\omega.q^{-1} = \begin{pmatrix} 0 \\ Q(q)\bar{\omega} \end{pmatrix}$$

with $\omega = (0, \bar{\omega}')'$, one gets

$$\begin{aligned} x.q_r.\omega_r.q_r^{-1} &= \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} . \begin{pmatrix} 0 \\ Q(q_r)\bar{\omega}_r \end{pmatrix} \\ &= \begin{pmatrix} -\bar{\omega}_r' Q(q_r)'\bar{x} \\ S(-Q(q_r)\bar{\omega}_r)\bar{x} + x_0 Q(q_r)\bar{\omega}_r \end{pmatrix} \end{aligned}$$

On the other hand

$$\omega_l.x = \begin{pmatrix} 0 \\ \bar{\omega}_l \end{pmatrix} . \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} = \begin{pmatrix} -\bar{\omega}_l'\bar{x} \\ S(\bar{\omega}_l)\bar{x} + x_0 \bar{\omega}_l \end{pmatrix}$$

Therefore

$$\frac{1}{2}(\omega_l.x + x.q_r.\omega_r.q_r^{-1}) = \begin{pmatrix} 0 & -\bar{\omega}' \\ \bar{\omega} & S(\bar{\bar{\omega}}) \end{pmatrix} \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix}$$

with

$$\begin{aligned} \bar{\omega} &= \frac{1}{2}(\bar{\omega}_l + Q(q_r)\bar{\omega}_r) \\ \bar{\bar{\omega}} &= \frac{1}{2}(\bar{\omega}_l - Q(q_r)\bar{\omega}_r) \end{aligned}$$

These relations specify the one-to-one correspondence between $(\bar{\omega}, \bar{\bar{\omega}})$ and $(\bar{\omega}_l, \bar{\omega}_r)$ from which (20) follows directly.
End of proof.
The practical usefulness of (20) is to allow for the on-line calculation of a time-varying rotation matrix $R(t) \in \mathbb{SO}(4)$ via the calculation of the corresponding pair of unitary quaternions $(q_l, q_r)(t)$, with the advantage of involving eight variables (i.e. four variables for each quaternion) instead of the sixteen components of the rotation matrix itself.