# Implementation of a Nonlinear Attitude Estimator for Aerial Robotic Vehicles 

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#### Abstract

Attitude estimation is a key component of the avionics suite of any aerial robotic vehicle. This paper details theoretical and practical solutions in order to obtain a robust nonlinear attitude estimator for flying vehicles equipped with low-cost sensors. The attitude estimator is based on a nonlinear explicit complementary filter that has been significantly enhanced with an effective gyro-bias compensation via the design of an antiwindup nonlinear integrator. A measurement decoupling strategy is proposed in order to make roll and pitch estimation robust to magnetic disturbances that are known to cause errors in yaw estimation. In addition, the paper discusses the fixed-point numerical implementation of the algorithm. Finally, simulation and experimental results confirm the performance of the proposed method.


Index Terms-attitude estimation, nonlinear observer, gyrobias compensation, anti-windup integrator, magnetic disturbance, unmanned aerial vehicle

## I. Introduction

The development of a small-scale low-cost autonomous aerial vehicle system requires effective solutions to a number of key technological problems. The avionics subsystem of such a vehicle is arguably the technology that is most closely coupled to the autonomy of the vehicle. Within an avionics system, the attitude estimator provides the primary measurement that ensures robust stability of the vehicle flight. The development of a robust and reliable attitude estimator, that can run on low-cost computational hardware, and that requires only measurements from low-cost and light-weight sensing systems, is a key technology enabler for the development of such systems. Theoretically, it is possible to estimate the attitude just by integrating the rigid body kinematic equation of rotation and using the angular velocity measurement data provided by gyrometers. However, such a solution is not viable beyond a few hundred milliseconds due to the effects of sensor noise and bias. Several alternative solutions have been proposed since the early 1960s'. The surveys about attitude estimation methods [8], [9], [11], [21] are useful references to begin research on the topic. In fact, the attitude can be algebraically determined by using the measurements, in the vehicle body-fixed frame, of at least two known noncollinear inertial directions (i.e. vectors) as explained in [24], [34], [40]. However, imperfect measurements and/or imprecise

[^0]knowledge of the considered inertial directions can generate large errors in the extracted attitude. This motivates the development of algorithms that fuse vector measurements, such as those provided by magnetometers or accelerometers, with the angular velocity measurements provided by gyrometers to obtain a more accurate and less noisy attitude estimation [3], [4], [5], [6], [7], [10], [12], [18], [20], [21], [22], [25], [26], [28], [29], [30], [31], [36], [37], [38], [39].

Recent advances in observer theory has lead to the development of nonlinear attitude observers that address this problem [4], [10], [21], [22], [26], [30], [37], [39]. These observers are algorithmically simple and can be implemented on lowprocessing power fixed-point microprocessors in unit quaternion form. The observers need only vector measurement inputs from low-cost and light-weight MEMS strap-down inertial measurement units (IMUs). Typically, the algorithms use a measurement of angular velocity measured by a 3-axis suite of rate gyrometers, a vector direction estimate of the gravitational direction derived from a 3-axis suite of accelerometers, and where possible, vector measurement of magnetic field, measured by a 3-axis suite of magnetometers [10], [21], [26]. All low-cost MEMS devices are subject to significant noise effects. Gyrometers and accelerometers suffer from time-varying bias and noise due to temperature change, vibration and impacts, magnetometer readings are corrupted by on-board magnetic fields generated by motors and currents, as well as external magnetic fields experienced by vehicles that manoeuvre in built environments. Earlier work in development of attitude observers tackled the question of bias in the gyrometer MEMS devices [10], [21], [26], [37], [38] by introducing an adaptive bias estimate in the algorithm. Decoupling of input signals to ensure that the roll and pitch estimates are not disturbed by deviation in the magnetometer measurements was considered in [14], [26] and represents an important modification of the basic algorithm to improve the overall quality of the attitude estimate. However, only local decoupling of the roll and pitch estimation from the magnetometer measurements has been proved [14], [26], the global decoupling has remained unsolved. Another question that has not been considered in the literature is that of limiting wind-up in the bias estimate of the algorithm. This is particularly important during start-up and take-off manoeuvres where the bias estimate integral can "wind-up" during the period when the vehicle is still in contact with the ground, potentially causing significant estimation errors at the moment the vehicle becomes airborne. There has also been little discussion of issues related to the fixed-point implementation of the algorithms on small-scale low-power microprocessors.

In the present paper we propose a novel nonlinear attitude estimation algorithm for IMUs that is almost-globally stable and locally exponentially stable, and that ensures the global decoupling of the dynamics of the roll and pitch estimates from magnetic disturbances and from the dynamics of the yaw estimate. Moreover, we propose an effective gyro-bias compensation via the design of an anti-windup nonlinear integrator. Finally, we discuss issues related to the fixedpoint implementation of the attitude estimation algorithm. The algorithm has been fully tested and experimental results provided indicate the performance of the algorithm.

## II. Problem Formulation

## A. The Explicit Complementary Filter for Attitude Estimation

The attitude of a rigid body can be modeled by a rotation between a body-fixed frame $\mathcal{B}$ and an inertial reference frame $\mathcal{I}$ (see, for example, [33], [35] for various attitude parameterizations). A convenient representation of attitude is the rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$, with $\mathrm{SO}(3)$ the special orthogonal group. The kinematic equation of $\mathbf{R}$ satisfies the following equation

$$
\begin{equation*}
\dot{\mathbf{R}}=\mathbf{R} \boldsymbol{\Omega}_{\times} \tag{1}
\end{equation*}
$$

with $\Omega \in \mathbb{R}^{3}$ the body's angular velocity expressed in the body-fixed frame $\mathcal{B}$, and $(\cdot)_{\times}$the skew-symmetric matrix associated with the cross product $\times$, i.e. $\mathbf{x}_{\times} \mathbf{y}=\mathbf{x} \times \mathbf{y}$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{3}$.

For analysis purposes, we use an Euler angle parametrization of the rotation matrices, knowing that singularities may occur for such a minimal parametrization. Let $\phi, \theta$ and $\psi$ denote the Euler angles corresponding to roll, pitch and yaw, commonly used in the aerospace field. Then, the attitude matrix $\mathbf{R}$ can be written as

$$
\mathbf{R}=\left[\begin{array}{ccc}
C \theta C \psi & S \phi S \theta C \psi-C \phi S \psi & C \phi S \theta C \psi+S \phi S \psi  \tag{2}\\
C \theta S \psi & S \phi S \theta S \psi+C \phi C \psi & C \phi S \theta S \psi-S \phi C \psi \\
-S \theta & S \phi C \theta & C \phi C \theta
\end{array}\right]
$$

with $C$ and $S$ denoting the $\cos (\cdot)$ and $\sin (\cdot)$ operators.
In practice, the angular velocity vector $\boldsymbol{\Omega}$ is typically measured by gyrometers. For the sake of observer design and associated analysis, the measured angular velocity, denoted as $\boldsymbol{\Omega}_{y}$, is modeled as the sum of the real angular velocity $\boldsymbol{\Omega}$ with an unknown constant (or slowly time-varying in practice) bias vector $\mathbf{b} \in \mathbb{R}^{3}$, i.e. $\boldsymbol{\Omega}_{y}=\boldsymbol{\Omega}+\mathbf{b}$ (see, for example, [21]).

Let us recall and discuss the explicit complementary filter proposed in [10], [21]. Let $\left\{\mathbf{v}_{i}^{\mathcal{I}}\right\}$ denotes a set of $n(\geq 2)$ known non-collinear unit vectors of coordinates expressed in the inertial frame $\mathcal{I}$, and $\left\{\mathbf{v}_{i}^{\mathcal{B}}\right\}$ a set of measurement data of these vectors expressed in the body-fixed frame $\mathcal{B}$. Let $\hat{\mathbf{R}}$ and $\hat{\mathbf{b}}$ denote the estimates of $\mathbf{R}$ and $\mathbf{b}$, respectively. The explicit complementary filter is written as

$$
\left\{\begin{align*}
\dot{\hat{\mathbf{R}}} & =\hat{\mathbf{R}}\left(\boldsymbol{\Omega}_{y}-\hat{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times},  \tag{3}\\
\dot{\hat{\mathbf{b}}} & =\sigma_{\mathbf{b}}, \quad \hat{\mathbf{b}}(0) \in \mathbb{R}^{3} \\
\sigma_{\mathbf{R}} & :=\sum_{i=1}^{n} k_{i} \mathbf{v}_{i}^{\mathcal{B}} \times \hat{\mathbf{R}}^{\top} \mathbf{v}_{i}^{I}, \quad \sigma_{\mathbf{b}}:=-k_{I} \sigma_{\mathbf{R}}
\end{align*}\right.
$$

with $k_{I}$ and $k_{i},(i=1, \cdots, n)$, denoting positive gains. As proved in [21], observer (3) ensures almost-global stability and
local exponential stability of the equilibrium $(\tilde{\mathbf{R}}, \tilde{\mathbf{b}})=\left(\mathbf{I}_{3}, \mathbf{0}\right)$, with $\tilde{\mathbf{R}}:=\mathbf{R} \hat{\mathbf{R}}^{\top}, \tilde{\mathbf{b}}:=\mathbf{b}-\hat{\mathbf{b}}$ and $\mathbf{I}_{3}$ the identity element of $\mathrm{SO}(3)$.

## B. Standard Implementation with IMUs

Assume that the IMU fixed to the body consists of a 3-axis gyrometer, a 3-axis accelerometer and a 3-axis magnetometer.

- The 3-axis accelerometer measures the specific acceleration $\mathbf{a}_{\mathcal{B}} \in \mathbb{R}^{3}$ expressed in the body-fixed frame $\mathcal{B}$. One has $\mathbf{a}_{\mathcal{B}}=\mathbf{R}^{\top}\left(\ddot{\mathbf{x}}-g \mathbf{e}_{3}\right)$, where the vehicle's acceleration expressed in the inertial frame $\mathcal{I}$ is $\ddot{\mathbf{x}}$, and the gravitational acceleration expressed in the frame $\mathcal{I}$ is $g \mathbf{e}_{3}$, with $\mathbf{e}_{3}=(0,0,1)^{\top}$.
It is known that for an ideal thrust controlled aerial vehicle, the measurement of the gravity direction cannot be directly extracted from accelerometer measurement data [23], [27]. In practice, aerial robotic systems are subject to secondary aerodynamic forces that inject low frequency information on the actual inertial gravitation direction into the accelerometer measurements [23], [27]. It follows that the model $\mathbf{a}_{B} \approx-g \mathbf{R}^{\top} \mathbf{e}_{3}$ is an effective model for vector attitude measurement in a wide range of practical systems [10], [20], [21], [26].
- The 3-axis magnetometer measures the geomagnetic field vector $\mathbf{m}_{\mathcal{B}} \in \mathbb{R}^{3}$ expressed in the body-fixed frame $\mathcal{B}$. If the magnetometer measurement data is not corrupted by magnetic disturbances, then one has $\mathbf{m}_{\mathcal{B}}=\mathbf{R}^{\top} \mathbf{m}_{\mathcal{I}}$, with $\mathbf{m}_{\mathcal{I}} \in \mathbb{R}^{3}$ the geomagnetic field expressed in the inertial frame $\mathcal{I}$. One can check at [1] for the geomagnetic field vector by using the IGRF-11 model.
Standard implementation of the explicit complementary filter (3) consists in defining the innovation term $\sigma_{\mathbf{R}}$ as (see [21] for more details)

$$
\begin{equation*}
\sigma_{\mathbf{R}}=k_{1}^{s} \mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{R}}^{\top} \mathbf{u}_{\mathcal{I}}+k_{2}^{s} \overline{\mathbf{m}}_{\mathcal{B}} \times \hat{\mathbf{R}}^{\top} \overline{\mathbf{m}}_{\mathcal{I}} \tag{4}
\end{equation*}
$$

with $k_{1,2}^{s}$ positive gains, $\mathbf{u}_{\mathcal{B}}:=-\mathbf{a}_{\mathcal{B}} / g, \mathbf{u}_{\mathcal{I}}:=\mathbf{e}_{3}, \overline{\mathbf{m}}_{\mathcal{B}}:=$ $\mathbf{m}_{\mathcal{B}} /\left|\mathbf{m}_{\mathcal{I}}\right|$ and $\overline{\mathbf{m}}_{\mathcal{I}}:=\mathbf{m}_{\mathcal{I}} /\left|\mathbf{m}_{\mathcal{I}}\right|$. Let us call this solution as standard observer to distinguish it with the conditioned observer proposed in Section III.

## C. Coupling Issues with Standard Implementation with IMUs

In view of Eq. (2) the roll and pitch angles $\phi$ and $\theta$ can be directly deduced from $\mathbf{R}^{\top} \mathbf{e}_{3}$ which can be approximated by the accelerometer measurement $\mathbf{a}_{\mathcal{B}}$. As a consequence, theoretically estimating roll and pitch can be done independently from magnetometer measurements. However, the standard implementation of the explicit complementary filter (3) with IMUs encounters some issues well discussed in the literature (see, for example, [26]):

- Magnetic disturbances and bias influence the estimation of roll and pitch angles. In many applications especially for small-size electric motorized aerial robots, significant magnetic disturbances are almost unavoidable, leading to significant time-varying deterministic error between $\mathbf{m}_{\mathcal{B}}$ and $\mathbf{R}^{\top} \mathbf{m}_{\mathcal{I}}$. This not only leads to large estimation errors of the yaw angle $\psi$ but also non-negligible errors in the roll and pitch estimation.
- The dynamics of roll, pitch and yaw estimates are highly coupled. This implies that the estimation of the yaw angle strongly affects the estimation of the roll and pitch angles. This issue can be observed when taking a close look at the linearized system around the system equilibrium. For the sake of simplicity, let us neglect the gyro-bias $\mathbf{b}$ and the dynamics of the estimated bias $\hat{\mathbf{b}}$ only in this subsection. This supposition in association with Eqs. (3) and (4) ensures the following dynamics of the error attitude $\tilde{\mathbf{R}}=\mathbf{R} \hat{\mathbf{R}}^{\top}$ :

$$
\begin{equation*}
\dot{\tilde{\mathbf{R}}}=-\left(k_{1}^{s} \mathbf{e}_{3} \times \tilde{\mathbf{R}} \mathbf{e}_{3}+k_{2}^{s} \overline{\mathbf{m}}_{\mathcal{I}} \times \tilde{\mathbf{R}} \overline{\mathbf{m}}_{\mathcal{I}}\right)_{\times} \tilde{\mathbf{R}} \tag{5}
\end{equation*}
$$

Consider a first order approximation of $\tilde{\mathbf{R}}$ around the equilibrium $\tilde{\mathbf{R}}=\mathbf{I}_{3}$ as $\tilde{\mathbf{R}}=\mathbf{I}_{3}+\mathbf{x}_{\times}$, with $\mathbf{x}=$ $\left(x_{1}, x_{2}, x_{3}\right)^{\top} \in \mathbb{R}^{3}$. Note that locally the first, second and third components of x correspond, respectively, to the roll, pitch and yaw error estimates. One easily verifies from Eq. (5) that

$$
\begin{align*}
& \dot{\mathbf{x}}=-k_{1}^{s} \mathbf{e}_{3} \times \mathbf{x}_{\times} \mathbf{e}_{3}-k_{2}^{s} \overline{\mathbf{m}}_{\mathcal{I}} \times \mathbf{x}_{\times} \overline{\mathbf{m}}_{\mathcal{I}} \\
& =\left(-k_{1}^{s}\left(\mathbf{I}_{3}-\mathbf{e}_{3} \mathbf{e}_{3}^{\top}\right)-k_{2}^{s}\left(\mathbf{I}_{3}-\overline{\mathbf{m}}_{\mathcal{I}} \overline{\mathbf{m}}_{\mathcal{I}}^{\top}\right)\right) \mathbf{x} \\
& =\left[\begin{array}{ccc}
-k_{1}^{s}-k_{2}^{s}\left(1-\bar{m}_{1}^{2}\right) & k_{2}^{s} \bar{m}_{1} \bar{m}_{2} & k_{2}^{s} \bar{m}_{1} \bar{m}_{3} \\
k_{2}^{s} \bar{m}_{1} \bar{m}_{2} & -k_{1}^{s}-k_{2}^{s}\left(1-\bar{m}_{2}^{2}\right) & k_{2}^{s} \bar{m}_{2} \bar{m}_{3} \\
k_{2}^{s} \bar{m}_{1} \bar{m}_{3} & k_{2}^{s} \bar{m}_{2} \bar{m}_{3} & -k_{2}^{s}\left(1-\bar{m}_{3}^{2}\right)
\end{array}\right] \mathbf{x} \\
& =\mathbf{A}_{\mathbf{x}} \mathbf{x} . \tag{6}
\end{align*}
$$

In practice, the gravity vector and the geomagnetic field vector (i.e. $\mathbf{e}_{3}$ and $\overline{\mathbf{m}}_{\mathcal{I}}$ ) can be "ill-conditioned" in the sense that they are very close to each other. In such a case, the third component of $\overline{\mathbf{m}}_{\mathcal{I}}$ is dominant to its first and second ones. For example, in France $\bar{m}_{3} \approx 0.9$. As a consequence, in view of Eq. (6) the dynamics of the roll and pitch errors (i.e. $x_{1}$ and $x_{2}$ ) are strongly coupled with the yaw error dynamics (i.e. $x_{3}$ ).

- On the other hand, the strong dynamics coupling is not the sole issue. The ill-conditioning of the two vectors $\mathbf{e}_{3}$ and $\overline{\mathbf{m}}_{\mathcal{I}}$ may also lead to the impossibility of finding a set of "non-high" gains $\left\{k_{1}^{s}, k_{2}^{s}\right\}$ so as to provide the system with fast time response, bearing in mind that high gains may excessively amplify measurement noises. For discussion purposes and without loss of generality, let us, for instance, assume that $\bar{m}_{2} \approx 0$ (i.e. $\bar{m}_{1}^{2}+\bar{m}_{3}^{2} \approx 1$ ) and $\bar{m}_{3}^{2} \gg \bar{m}_{1}^{2}$. Under this approximation, it is straightforward to verify that three poles of System (6) are given by:

$$
\left\{\begin{array}{l}
\lambda_{1}^{s}=-\left(k_{1}^{s}+k_{2}^{s}\right)  \tag{7}\\
\lambda_{2}^{s}=-\frac{1}{2}\left(k_{1}^{s}+k_{2}^{s}\right)\left(1+\sqrt{1-\frac{4 k_{1}^{s} k_{2}^{s} \bar{m}_{1}^{2}}{\left(k_{1}^{s}+k_{2}^{s}\right)^{2}}}\right) \\
\lambda_{3}^{s}=-\frac{1}{2}\left(k_{1}^{s}+k_{2}^{s}\right)\left(1-\sqrt{1-\frac{4 k_{1}^{s} k_{2}^{s} \bar{m}_{1}^{2}}{\left(k_{1}^{s}+k_{2}^{s}\right)^{2}}}\right) \approx-\frac{k_{1}^{s} k_{2}^{s} \bar{m}_{1}^{2}}{k_{1}^{s}+k_{2}^{s}}
\end{array}\right.
$$

The pole $\lambda_{1}^{s}$ is associated with the pitch dynamics, and the poles $\lambda_{2}^{s}$ and $\lambda_{3}^{s}$ are associated with the coupled roll and yaw dynamics. The less negative pole $\lambda_{3}^{s}$, approximated by $-k_{1}^{s} k_{2}^{s} \bar{m}_{1}^{2} /\left(k_{1}^{s}+k_{2}^{s}\right)$, will be very close to zero if $k_{1}^{s}$ and $k_{2}^{s}$ are not chosen sufficiently high, since $\bar{m}_{1}^{2} \ll 1$. This leads to slow time response of the coupled roll and yaw dynamics.

In Section III we propose a novel observer ensuring the global decoupling of the roll and pitch estimations from the yaw estimation and from magnetometer measurements.

## D. Wind-up Issues on Gyro-bias Estimation

Additionally to the decoupling issues, one can observe some problems in practice when using the integral correction term b , meant to compensate for the unknown constant bias vector b. For instance, the integral term $\hat{\mathbf{b}}$ may grow arbitrarily large leading to slow desaturation (and slow convergence) and/or important overshoots of the estimation error variables (i.e. $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{b}}$ ). Various sources of this phenomenon can be identified such as large initial errors, poor gain tuning, imprecise knowledge of the considered inertial vectors, and imperfect measurements of these vectors due to sensor misalignment, sensor vibration, noises and biases, etc. Consequently, one may choose a very small value for the integral gain $k_{I}$ to reduce overshoots, but this in turn degrades the estimation performance. In fact, this issue is well-known as the so-called "integral wind-up effects" and has been well studied in the literature, particularly in the context of control systems [13], [15], [17], [32]. However, in the context of attitude estimation design, to our knowledge, there are no standard references for anti-windup for observers.
In Section III we propose some modifications on the dynamics of $\hat{\mathbf{b}}$ so that the properties of convergence and stability of the filter are still ensured, and integral wind-up effects can be limited.

## III. Observer Design for IMUs

Let us make the approximation that $\mathbf{a}_{\mathcal{B}} \approx-g \mathbf{R}^{\top} \mathbf{e}_{3}$ and compute the following vectors (see Figure 1)

$$
\left\{\begin{align*}
\mathbf{u}_{\mathcal{B}}:=-\frac{\mathbf{a}_{\mathcal{B}}}{g}, & \mathbf{v}_{\mathcal{B}}:=\frac{\pi_{\mathbf{u}_{\mathcal{B}}} \mathbf{m}_{\mathcal{B}}}{\left|\pi_{\mathbf{u}_{\mathcal{I}}} \mathbf{m}_{\mathcal{I}}\right|}  \tag{8}\\
\mathbf{u}_{\mathcal{I}}:=\mathbf{e}_{3}, & \mathbf{v}_{\mathcal{I}}:=\frac{\pi_{\mathbf{u}_{\mathcal{I}}} \mathbf{m}_{\mathcal{I}}}{\left|\pi_{\mathbf{u}_{\mathcal{I}}} \mathbf{m}_{\mathcal{I}}\right|}
\end{align*}\right.
$$

with $\pi_{\mathbf{x}}:=|\mathbf{x}|^{2} \mathbf{I}_{3}-\mathbf{x} \mathbf{x}^{\top}, \forall \mathbf{x} \in \mathbb{R}^{3}$, denoting the orthogonal projection on the plan orthogonal to $\mathbf{x}$. One easily verifies that

$$
\begin{equation*}
\mathbf{u}_{\mathcal{B}}=\mathbf{R}^{\top} \mathbf{u}_{\mathcal{I}}, \quad \mathbf{v}_{\mathcal{B}}=\mathbf{R}^{\top} \mathbf{v}_{\mathcal{I}} \tag{9}
\end{equation*}
$$

Denote also

$$
\begin{equation*}
\hat{\mathbf{u}}_{\mathcal{B}}:=\hat{\mathbf{R}}^{\top} \mathbf{u}_{\mathcal{I}}, \quad \hat{\mathbf{v}}_{\mathcal{B}}:=\hat{\mathbf{R}}^{\top} \mathbf{v}_{\mathcal{I}} . \tag{10}
\end{equation*}
$$

Theorem 1: Consider the rotation kinematics (1) and the angular velocity measurement model

$$
\begin{equation*}
\boldsymbol{\Omega}_{y}=\boldsymbol{\Omega}+\mathbf{b} \tag{11}
\end{equation*}
$$

with $\mathbf{b} \in \mathbb{R}^{3}$ an unknown constant bias. Consider the following "conditioned observer":

$$
\left\{\begin{align*}
\dot{\hat{\mathbf{R}}} & =\hat{\mathbf{R}}\left(\boldsymbol{\Omega}_{y}-\hat{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times}, \quad \hat{\mathbf{R}}(0) \in \mathrm{SO}(3)  \tag{12}\\
\dot{\hat{\mathbf{b}}} & =-k_{b} \hat{\mathbf{b}}+k_{b} \operatorname{sat}_{\Delta}(\hat{\mathbf{b}})+\sigma_{\mathbf{b}}, \quad|\hat{\mathbf{b}}(0)|<\Delta \\
\sigma_{\mathbf{R}} & :=k_{1} \mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}+k_{2} \hat{\mathbf{u}}_{\mathcal{B}} \hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right) \\
\sigma_{\mathbf{b}} & :=-k_{3} \mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}-k_{4} \mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}
\end{align*}\right.
$$



Fig. 1. Vectors involved in attitude estimation.
with $k_{1}, k_{2}, k_{3}, k_{4}, k_{b}$ and $\Delta$ denoting positive numbers, and sat $_{\Delta}(\cdot)$ the classical saturation function defined by sat ${ }_{\Delta}(\mathbf{x}):=$ $\mathbf{x} \min (1, \Delta /|\mathbf{x}|)$. Choose $k_{3}$ and $k_{4}$ such that

$$
\begin{equation*}
k_{4}<k_{3} \tag{13}
\end{equation*}
$$

Assume that $\Omega$ is bounded and that the gyro-bias $\mathbf{b}$ is bounded in norm by $\Delta$, i.e. $|\mathbf{b}| \leq \Delta$. Then,

1) The dynamics of the estimate errors $(\tilde{\mathbf{R}}, \tilde{\mathbf{b}})$, with $\tilde{\mathbf{R}}=$ $\mathbf{R} \hat{\mathbf{R}}_{\tilde{\mathbf{b}}}{ }^{\top}$ and $\tilde{\mathbf{b}}=\mathbf{b}-\hat{\mathbf{b}}$, have only four isolated equilibria $(\tilde{\mathbf{R}}, \tilde{\mathbf{b}})=\left(\tilde{\mathbf{R}}_{i}^{\star}, \mathbf{0}\right), \quad \underset{\tilde{\mathbf{R}}}{ } \quad i=0, \cdots, 3$, with $\tilde{\mathbf{R}}_{0}^{\star}=\mathbf{I}_{3}$.
2) The equilibrium $(\tilde{\mathbf{R}}, \tilde{\mathbf{b}})=\left(\mathbf{I}_{3}, \mathbf{0}\right)$ is locally exponentially stable.
3) The equilibria $\left(\tilde{\mathbf{R}}_{1}^{\star}, \mathbf{0}\right),\left(\tilde{\mathbf{R}}_{2}^{\star}, \mathbf{0}\right)$ and $\left(\tilde{\mathbf{R}}_{3}^{\star}, \mathbf{0}\right)$ are unstable. Thus, for almost all initial conditions $(\hat{\mathbf{R}}(0), \hat{\mathbf{b}}(0)) \neq\left(\tilde{\mathbf{R}}_{i}^{\star \top} \mathbf{R}(0), \mathbf{b}\right), \quad i=1,2,3$, the trajectory $(\hat{\mathbf{R}}(t), \hat{\mathbf{b}}(t))$ converges to the trajectory $(\mathbf{R}(t), \mathbf{b}(t))$.
4) The dynamics of $\hat{\mathbf{u}}_{\mathcal{B}}$ does not depend on $\mathbf{m}_{\mathcal{B}}$ when considering $\hat{\mathbf{b}}$ as an input.
5) The estimated gyro-bias $\hat{\mathbf{b}}$ is bounded in norm by $\bar{\Delta}:=$ $\Delta+\left(k_{3}+k_{4}\right) / k_{b}$, i.e. $|\hat{\mathbf{b}}(t)| \leq \bar{\Delta}, \forall t \geq 0$.
Proof: It is straightforward to prove Parts 4 and 5 of the theorem statement. Using (10) and the first equation of (12), one deduces

$$
\begin{aligned}
\dot{\hat{\mathbf{u}}}_{\mathcal{B}} & =-\left(\boldsymbol{\Omega}_{y}-\hat{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times} \hat{\mathbf{u}}_{\mathcal{B}} \\
& =-\left(\boldsymbol{\Omega}_{y}-\hat{\mathbf{b}}+k_{1} \mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)_{\times} \hat{\mathbf{u}}_{\mathcal{B}}
\end{aligned}
$$

where the last equality is obtained using

$$
\left(k_{2} \hat{\mathbf{u}}_{\mathcal{B}} \hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right)\right)_{\times} \hat{\mathbf{u}}_{\mathcal{B}}=\mathbf{0}
$$

From here, it is clear that the dynamics of $\hat{\mathbf{u}}_{\mathcal{B}}$ is independent of the magnetometer measurement vector $\mathbf{m}_{\mathcal{B}}$ when considering $\hat{\mathbf{b}}$ as an input.

The proof of Part 5 of the theorem statement is based on the positive function $\mathcal{V}=1 / 2|\hat{\mathbf{b}}|^{2}$ whose time-derivative verifies

$$
\begin{align*}
\dot{\mathcal{V}} & =-k_{b}|\hat{\mathbf{b}}|^{2}+\hat{\mathbf{b}}^{\top}\left(k_{b} \operatorname{sat}_{\Delta}(\hat{\mathbf{b}})+\sigma_{\mathbf{b}}\right) \\
& \leq-k_{b}|\hat{\mathbf{b}}|^{2}+|\hat{\mathbf{b}}|\left(k_{b} \Delta+\sup \left(\sigma_{\mathbf{b}}\right)\right) \\
& \leq-k_{b}|\hat{\mathbf{b}}|^{2}+k_{b}|\hat{\mathbf{b}}|\left(\Delta+\frac{k_{3}+k_{4}}{k_{b}}\right) . \tag{14}
\end{align*}
$$

Let us prove by contradiction that $|\hat{\mathbf{b}}(t)| \leq \bar{\Delta}=\Delta+\left(k_{3}+\right.$ $\left.k_{4}\right) / k_{b}, \forall t \geq 0$. Assume there exist a time instant $T>0$ and a small number $\varepsilon>0$ such that $|\hat{\mathbf{b}}(T)|>\bar{\Delta},|\hat{\mathbf{b}}(T-\varepsilon)|=\bar{\Delta}$ and $|\hat{\mathbf{b}}(\tau)|$ is increasing for $\tau \in[T-\varepsilon, T]$. This implies that $\mathcal{V}(\tau)=1 / 2|\hat{\mathbf{b}}(\tau)|^{2}$ is also increasing for $\tau \in[T-\varepsilon, T]$. This implies that $\dot{\mathcal{V}}(T-\varepsilon / 2)>0$. However, since $|\hat{\mathbf{b}}(T-\varepsilon / 2)|>$ $|\hat{\mathbf{b}}(T-\varepsilon)|=\bar{\Delta}$, inequality (14) implies that $\dot{\mathcal{V}}(T-\varepsilon / 2)<0$. The resulting contradiction allows one to conclude the proof of Part 5 of the theorem statement.

We consider the following candidate Lyapunov function

$$
\begin{equation*}
\mathcal{L}:=\left(1-\mathbf{u}_{\mathcal{B}}^{\top} \hat{\mathbf{u}}_{\mathcal{B}}\right)+\frac{k_{4}}{k_{3}}\left(1-\mathbf{v}_{\mathcal{B}}^{\top} \hat{\mathbf{v}}_{\mathcal{B}}\right)+\frac{1}{2 k_{3}}|\tilde{\mathbf{b}}|^{2} \tag{15}
\end{equation*}
$$

and compute its time-derivative. First, one verifies from Eqs. (9), (10), (11) and (12) that

$$
\begin{aligned}
\frac{d}{d t}\left(1-\mathbf{u}_{\mathcal{B}}^{\top} \hat{\mathbf{u}}_{\mathcal{B}}\right)= & -\mathbf{u}_{\mathcal{B}}^{\top} \boldsymbol{\Omega}_{\times} \hat{\mathbf{u}}_{\mathcal{B}}+\mathbf{u}_{\mathcal{B}}^{\top}\left(\boldsymbol{\Omega}+\tilde{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times} \hat{\mathbf{u}}_{\mathcal{B}} \\
= & -k_{1}\left|\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right|^{2}-\tilde{\mathbf{b}}^{\top}\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right) \\
\frac{d}{d t}\left(1-\mathbf{v}_{\mathcal{B}}^{\top} \hat{\mathbf{v}}_{\mathcal{B}}\right)= & -\mathbf{v}_{\mathcal{B}}^{\top} \boldsymbol{\Omega}_{\times} \hat{\mathbf{v}}_{\mathcal{B}}+\mathbf{v}_{\mathcal{B}}^{\top}\left(\boldsymbol{\Omega}+\tilde{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times} \hat{\mathbf{v}}_{\mathcal{B}} \\
= & -k_{1}\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right) \\
& -k_{2}\left(\hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right)\right)^{2}-\tilde{\mathbf{b}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right), \\
\dot{\tilde{\mathbf{b}}}=-\dot{\hat{\mathbf{b}}}= & -k_{b} \tilde{\mathbf{b}}^{2}+k_{b}\left(\mathbf{b}-\operatorname{sat}_{\Delta}(\mathbf{b}-\tilde{\mathbf{b}})\right)-\sigma_{\mathbf{b}}
\end{aligned}
$$

From here, one straightforwardly verifies that the timederivative of $\mathcal{L}$ is given by

$$
\begin{align*}
\dot{\mathcal{L}}= & -k_{1}\left|\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right|^{2}-\frac{k_{2} k_{4}}{k_{3}}\left(\hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right)\right)^{2} \\
& -\frac{k_{1} k_{4}}{k_{3}}\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right) \\
& \left.-\frac{k_{b}}{k_{3}}|\tilde{\mathbf{b}}|^{2}+\frac{k_{b}}{k_{3}} \tilde{\mathbf{b}}^{\top}\left(\mathbf{b}-\operatorname{sat}_{\Delta}(\mathbf{b}-\tilde{\mathbf{b}})\right)\right) . \tag{16}
\end{align*}
$$

Then, using the inequality $\left|\mathbf{b}-\operatorname{sat}_{\Delta}(\mathbf{b}-\tilde{\mathbf{b}})\right| \leq|\tilde{\mathbf{b}}|, \forall \tilde{\mathbf{b}} \in \mathbb{R}^{3}$, provided that $\Delta \geq|\mathbf{b}|$ by assumption (see [11, Chap.2, Sec. 2.8.14] for the proof), one deduces from Eq. (16) that

$$
\begin{align*}
\dot{\mathcal{L}} \leq & -k_{1}\left|\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right|^{2}-\frac{k_{2} k_{4}}{k_{3}}\left(\hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right)\right)^{2} \\
& -\frac{k_{1} k_{4}}{k_{3}}\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right) . \tag{17}
\end{align*}
$$

In the sequel, relation (17) will be further developed. From the definition of $\mathbf{v}_{\mathcal{I}}$ given in (8), one deduces that this constant unit vector is orthogonal to $\mathbf{e}_{3}$ and, consequently, belongs to $\operatorname{Span}\left(\mathbf{e}_{1}, \mathbf{e}_{2}\right)$. Thus, there exists a constant angle $\alpha$ such that

$$
\mathbf{v}_{\mathcal{I}}=C \alpha \mathbf{e}_{1}+S \alpha \mathbf{e}_{2}=\underbrace{\left[\begin{array}{ccc}
C \alpha & -S \alpha & 0  \tag{18}\\
S \alpha & C \alpha & 0 \\
0 & 0 & 1
\end{array}\right]}_{=: \mathbf{R}_{\alpha} \in \operatorname{SO}(3)} \mathbf{e}_{1}=\mathbf{R}_{\alpha} \mathbf{e}_{1}
$$

Define $\underline{\mathbf{R}}:=\mathbf{R}_{\alpha}^{\top} \mathbf{R}, \underline{\hat{\mathbf{R}}}:=\mathbf{R}_{\alpha}^{\top} \hat{\mathbf{R}}$, and the new attitude error $\underline{\tilde{\mathbf{R}}}:=\underline{\mathbf{R}} \underline{\mathbf{R}}^{\top}$. One verifies that $\tilde{\mathbf{R}}=\mathbf{R}_{\alpha} \underline{\tilde{\mathbf{R}}} \mathbf{R}_{\alpha}^{\top}$. Consequently, if $\underline{\tilde{\mathbf{R}}}$ converges to $\mathbf{I}_{3}$, then so does $\tilde{\mathbf{R}}$. Using the fact that $\mathbf{u}_{\mathcal{I}}=\mathbf{e}_{3}=\mathbf{R}_{\alpha} \mathbf{e}_{3}$ and $\mathbf{v}_{\mathcal{I}}=\mathbf{R}_{\alpha} \mathbf{e}_{1}$, one verifies the following relations:
$\mathbf{u}_{\mathcal{B}}=\underline{\mathbf{R}}^{\top} \mathbf{e}_{3}, \mathbf{v}_{\mathcal{B}}=\underline{\mathbf{R}}^{\top} \mathbf{e}_{1}, \hat{\mathbf{u}}_{\mathcal{B}}=\underline{\hat{\mathbf{R}}}^{\top} \mathbf{e}_{3}, \hat{\mathbf{v}}_{\mathcal{B}}=\underline{\hat{\mathbf{R}}}^{\top} \mathbf{e}_{1}$.

Let $\mathbb{Q}$ denote the group of unit quaternions. Denote the unit quaternion associated with $\underline{\tilde{\mathbf{R}}}$ as $\tilde{\mathbf{q}}:=\left(\tilde{q}_{0}, \tilde{\mathbf{q}}_{v}\right)^{\top} \in \mathbb{Q}$, where $\tilde{q}_{0}$ and $\tilde{\mathbf{q}}_{v}=\left(\tilde{q}_{1}, \tilde{q}_{2}, \tilde{q}_{3}\right)^{\top}$ are its real and imaginary parts, respectively. Then, using the Rodrigues' rotation formula

$$
\underline{\tilde{\mathbf{R}}}=\left(\tilde{q}_{0}^{2}-\left|\tilde{\mathbf{q}}_{v}\right|^{2}\right) \mathbf{I}_{3}+2 \tilde{q}_{0} \tilde{\mathbf{q}}_{v \times}+2 \tilde{\mathbf{q}}_{v} \tilde{\mathbf{q}}_{v}^{\top}
$$

along with Eq. (19), one verifies that

$$
\begin{aligned}
\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}} & =\underline{\hat{\mathbf{R}}}^{\top}\left(\underline{\tilde{\mathbf{R}}}^{\top} \mathbf{e}_{3} \times \mathbf{e}_{3}\right) \\
& =2 \underline{\hat{\mathbf{R}}}^{\top}\left(\left(-\tilde{q}_{0} \tilde{\mathbf{q}}_{v \times}+\tilde{\mathbf{q}}_{v} \tilde{\mathbf{q}}_{v}^{\top}\right) \mathbf{e}_{3} \times \mathbf{e}_{3}\right) \\
& =2 \underline{\hat{\mathbf{R}}}^{\top}\left(-\tilde{q}_{0} \mathbf{e}_{3 \times}^{2} \tilde{\mathbf{q}}_{v}-\tilde{q}_{3} \mathbf{e}_{3} \times \tilde{\mathbf{q}}_{v}\right) \\
& =2 \underline{\hat{\mathbf{R}}}^{\top}\left(\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{3} \mathbf{e}_{3}-\tilde{q}_{3} \mathbf{e}_{3} \times \tilde{\mathbf{q}}_{v}\right)
\end{aligned}
$$

Similarly, one obtains

$$
\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}=2 \underline{\hat{\mathbf{R}}}^{\top}\left(\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{1} \mathbf{e}_{1}-\tilde{q}_{1} \mathbf{e}_{1} \times \tilde{\mathbf{q}}_{v}\right)
$$

From here, one deduces that

$$
\begin{align*}
\left|\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right|^{2} & =4\left|\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{3} \mathbf{e}_{3}-\tilde{q}_{3} \mathbf{e}_{3} \times \tilde{\mathbf{q}}_{v}\right|^{2} \\
& =4\left(\tilde{q}_{0}^{2}\left|\tilde{\mathbf{q}}_{v}\right|^{2}-\tilde{q}_{0}^{2} \tilde{q}_{3}^{2}+\tilde{q}_{3}^{2}\left|\mathbf{e}_{3} \times \tilde{\mathbf{q}}_{v}\right|^{2}\right) \\
& =4\left(\tilde{q}_{0}^{2}+\tilde{q}_{3}^{2}\right)\left(\tilde{q}_{1}^{2}+\tilde{q}_{2}^{2}\right),  \tag{20}\\
\left(\hat{\mathbf{u}}_{\mathcal{B}}^{\top}\left(\mathbf{v}_{\mathcal{B}} \times \hat{\mathbf{v}}_{\mathcal{B}}\right)\right)^{2} & =4\left(\mathbf{e}_{3}^{\top}\left(\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{1} \mathbf{e}_{1}-\tilde{q}_{1} \mathbf{e}_{1} \times \tilde{\mathbf{q}}_{v}\right)\right)^{2} \\
& =4\left(\tilde{q}_{0} \tilde{q}_{3}-\tilde{q}_{1} \tilde{q}_{2}\right)^{2}, \tag{21}
\end{align*}
$$

and

$$
\begin{align*}
& \left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)^{\top}\left(\mathbf{v} \mathcal{B} \times \hat{\mathbf{v}}_{\mathcal{B}}\right) \\
& =4\left(\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{3} \mathbf{e}_{3}-\tilde{q}_{3} \mathbf{e}_{3 \times} \tilde{\mathbf{q}}_{v}\right)^{\top}\left(\tilde{q}_{0} \tilde{\mathbf{q}}_{v}-\tilde{q}_{0} \tilde{q}_{1} \mathbf{e}_{1}-\tilde{q}_{1} \mathbf{e}_{1 \times} \tilde{\mathbf{q}}_{v}\right) \\
& =4\left(\tilde{q}_{0}^{2} \tilde{q}_{2}^{2}+\tilde{q}_{1} \tilde{q}_{3}\left(e_{3} \times \tilde{\mathbf{q}}_{v}\right)^{\top}\left(e_{1} \times \tilde{\mathbf{q}}_{v}\right)\right) \\
& =4\left(\tilde{q}_{0}^{2} \tilde{q}_{2}^{2}-\tilde{q}_{1}^{2} \tilde{q}_{3}^{2}\right) \tag{22}
\end{align*}
$$

The substitution of Eqs. (20), (21) and (22) into (17) yields

$$
\begin{align*}
\dot{\mathcal{L}} \leq & -\frac{4 k_{2} k_{4}}{k_{3}}\left(\tilde{q}_{0} \tilde{q}_{3}-\tilde{q}_{1} \tilde{q}_{2}\right)^{2}-4 k_{1}\left(\tilde{q}_{0}^{2} \tilde{q}_{1}^{2}+\tilde{q}_{2}^{2} \tilde{q}_{3}^{2}\right) \\
& -4 k_{1}\left(1+\frac{k_{4}}{k_{3}}\right) \tilde{q}_{0}^{2} \tilde{q}_{2}^{2}-4 k_{1}\left(1-\frac{k_{4}}{k_{3}}\right) \tilde{q}_{1}^{2} \tilde{q}_{3}^{2} \tag{23}
\end{align*}
$$

Condition (13) and inequality (23) ensure the non-positivity of $\dot{\mathcal{L}}$ and, consequently, the non-increasing of $\mathcal{L}$. This and the definition (15) of $\mathcal{L}$ ensure the boundedness of $\tilde{\mathbf{b}}$ and, thus, of $\dot{\tilde{\mathbf{b}}}$. Then, one can easily verify from Eq. (16) that $\ddot{\mathcal{L}}$ also remains bounded, which implies the uniform continuity of $\dot{\mathcal{L}}$. From here, the application of Barbalat's lemma (see [16]) ensures the convergence of $\dot{\mathcal{L}}$ to zero. This convergence and inequality (23) imply the following relations:

$$
\left\{\begin{array}{l}
\tilde{q}_{0} \tilde{q}_{1} \rightarrow 0 \\
\tilde{q}_{0} \tilde{q}_{2} \rightarrow 0 \\
\tilde{q}_{1} \tilde{q}_{3} \rightarrow 0 \\
\tilde{q}_{2} \tilde{q}_{3} \rightarrow 0 \\
\tilde{q}_{0} \tilde{q}_{3} \rightarrow \tilde{q}_{1} \tilde{q}_{2}
\end{array}\right.
$$

which ensure the convergence of the unit quaternion $\tilde{\mathbf{q}}$ associated with $\underline{\tilde{\mathbf{R}}}$ to one of the following quaternions:

$$
\begin{array}{ll}
( \pm 1,(0,0,0))^{\top}, & (0,( \pm 1,0,0))^{\top} \\
(0,(0, \pm 1,0))^{\top}, & (0,(0,0, \pm 1))^{\top}
\end{array}
$$

which respectively correspond to the following rotation matrices:

$$
\left\{\begin{array}{l}
\underline{\tilde{\mathbf{R}}}_{0}^{\star}:=\mathbf{I}_{3}  \tag{24}\\
\underline{\tilde{\mathbf{R}}}_{1}^{\star}:=\operatorname{diag}([1,-1,-1]) \\
\tilde{\tilde{\mathbf{R}}}_{2}^{\star}:=\operatorname{diag}([-1,1,-1]) \\
\underline{\tilde{\mathbf{R}}}_{3}^{\star}:=\operatorname{diag}([-1,-1,1])
\end{array}\right.
$$

One verifies that $\tilde{\mathbf{R}}_{i}^{\star}=\mathbf{R}_{\alpha} \underline{\mathbf{R}}_{i}^{\star} \mathbf{R}_{\alpha}^{\top}$, with $i \in\{0,1,2,3\}$, and, thus, $\tilde{\mathbf{R}}_{0}^{\star}=\mathbf{I}_{3}$. It remains to prove that $\tilde{\mathbf{b}}$ converges to the null vector. To this end, we analyze the dynamics associated with $\underline{\tilde{\mathbf{R}}}$ :

$$
\begin{align*}
\underline{\dot{\mathbf{R}}} & =\underline{\mathbf{R}} \boldsymbol{\Omega}_{\times} \underline{\hat{\mathbf{R}}}^{\top}-\underline{\mathbf{R}}\left(\boldsymbol{\Omega}+\tilde{\mathbf{b}}+\sigma_{\mathbf{R}}\right)_{\times} \underline{\hat{\mathbf{R}}}^{\top} \\
& =-\left(\overline{\mathbf{b}}+k_{1} \mathbf{e}_{3} \times \underline{\tilde{\mathbf{R}}} \mathbf{e}_{3}+k_{2} \underline{\tilde{\mathbf{R}}} \mathbf{e}_{3} \mathbf{e}_{3}^{\top}\left(\underline{\tilde{\mathbf{R}}}^{\top} \mathbf{e}_{1} \times \mathbf{e}_{1}\right)\right) \times \underline{\tilde{\mathbf{R}}} \\
& =-\left(\overline{\mathbf{b}}+k_{1} \mathbf{e}_{3} \times \underline{\tilde{\mathbf{R}}} \mathbf{e}_{3}-k_{2} \underline{\tilde{\mathbf{R}}} \mathbf{e}_{3}\left(\mathbf{e}_{1}^{\top} \underline{\tilde{\mathbf{R}}} \mathbf{e}_{2}\right)\right)_{\times} \underline{\tilde{\mathbf{R}}} \tag{25}
\end{align*}
$$

with $\overline{\mathbf{b}}:=\underline{\mathbf{R}} \tilde{\mathbf{b}}$. Since $\underline{\tilde{\mathbf{R}}}$ converges to a constant matrix $\underline{\underline{\mathbf{R}}}_{i}^{\star}$ (as proved previously) and $\underline{\tilde{\mathbf{R}}}$ is uniformly continuous, one ensures that $\underline{\dot{\mathbf{R}}}$ converges to the null matrix. With $\underline{\tilde{\mathbf{R}}}_{i}$ specified by (24) one verifies that $\mathbf{e}_{3} \times \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{3}=\mathbf{e}_{1}^{\top} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{2}=\mathbf{0}$. Therefore, one ensures that $\overline{\mathbf{b}}$ and, thus, $\tilde{\mathbf{b}}$ converge to the null vector.

We proceed by computing the dynamics of the new variable $\overline{\mathbf{b}}$ and using the dynamics of $(\underline{\tilde{\mathbf{R}}}, \overline{\mathbf{b}})$ to prove the stability properties of the equilibria. The dynamics of $\bar{b}$ are locally given by

$$
\begin{align*}
\dot{\overline{\mathbf{b}}} & =\underline{\tilde{\mathbf{R}}} \boldsymbol{\Omega}_{\times} \tilde{\mathbf{b}}-\underline{\mathbf{R}} \sigma_{\mathbf{b}} \\
& =(\underline{\boldsymbol{\Omega}})_{\times} \overline{\mathbf{b}}+k_{3} \mathbf{e}_{3} \times \underline{\tilde{\mathbf{R}}} \mathbf{e}_{3}+k_{4} \mathbf{e}_{1} \times \underline{\tilde{\mathbf{R}}} \mathbf{e}_{1} \tag{26}
\end{align*}
$$

with $\underline{\Omega}:=\underline{\mathbf{R}} \Omega$. Consider a first order approximation of $(\underline{\tilde{\mathbf{R}}}, \overline{\mathbf{b}})$ (25) and (26) around an equilibrium point $\left(\underline{\tilde{\mathbf{R}}}_{i}^{\star}, \mathbf{0}\right)$, with $i \in$ $\{0,1,2,3\}$,

$$
\underline{\tilde{\mathbf{R}}}=\underline{\tilde{\mathbf{R}}}_{i}^{\star}\left(\mathbf{I}_{3}+\mathbf{x}_{\times}\right), \quad \overline{\mathbf{b}}=-\mathbf{y}, \quad \text { with } \mathbf{x}, \mathbf{y} \in \mathbb{R}^{3}
$$

The linearization of Eq. (25) is given by

$$
\underline{\tilde{\mathbf{R}}}_{i}^{\star} \dot{\mathbf{x}}_{\times}=\left(\mathbf{y}-k_{1} \mathbf{e}_{3 \times} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{3}+k_{2} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{3}\left(\mathbf{e}_{1}^{\top} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{2}\right)\right)_{\times} \underline{\tilde{\mathbf{R}}}_{i}^{\star}
$$

and, thus,

$$
\begin{align*}
\dot{\mathbf{x}} & =-k_{1} \tilde{\mathbf{R}}_{i}^{\star} \mathbf{e}_{3 \times} \tilde{\mathbf{R}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{3}+k_{2} \mathbf{e}_{3}\left(\mathbf{e}_{1}^{\top} \tilde{\mathbf{R}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{2}\right)+\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{y} \\
& =k_{1}\left(\tilde{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{3}\right)_{\times} \mathbf{e}_{3 \times} \mathbf{x}-k_{2} \mathbf{e}_{3}\left(\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{1} \times \mathbf{e}_{2}\right)^{\top} \mathbf{x}+\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{y} \\
& =\mathbf{A}_{i} \mathbf{x}+\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{y} \tag{27}
\end{align*}
$$

with

$$
\left\{\begin{array}{l}
\mathbf{A}_{0}:=\operatorname{diag}\left(\left[-k_{1},-k_{1},-k_{2}\right]\right)  \tag{28}\\
\mathbf{A}_{1}:=\operatorname{diag}\left(\left[k_{1}, k_{1},-k_{2}\right]\right) \\
\mathbf{A}_{2}:=\operatorname{diag}\left(\left[k_{1}, k_{1}, k_{2}\right]\right) \\
\mathbf{A}_{3}:=\operatorname{diag}\left(\left[-k_{1},-k_{1}, k_{2}\right]\right)
\end{array}\right.
$$

The linearization of Eq. (26) can be written as

$$
\begin{align*}
\dot{\mathbf{y}} & =-k_{3} \mathbf{e}_{3 \times} \times \tilde{\mathbf{R}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{3}-k_{4} \mathbf{e}_{1 \times} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}_{\times} \mathbf{e}_{1}+\underline{\boldsymbol{\Omega}}_{\times} \mathbf{y} \\
& =k_{3} \mathbf{e}_{3 \times}\left(\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{3}\right)_{\times} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}+k_{4} \mathbf{e}_{1 \times}\left(\underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{e}_{1}\right)_{\times} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}+\underline{\boldsymbol{\Omega}}_{\times} \mathbf{y} \\
& =\mathbf{B}_{i} \mathbf{x}+\underline{\boldsymbol{\Omega}}_{\times} \mathbf{y} \tag{29}
\end{align*}
$$

with

$$
\left\{\begin{array}{l}
\mathbf{B}_{0}:=\operatorname{diag}\left(\left[-k_{3},-k_{3}-k_{4},-k_{4}\right]\right)  \tag{30}\\
\mathbf{B}_{1}:=\operatorname{diag}\left(\left[k_{3},-k_{3}+k_{4}, k_{4}\right]\right) \\
\mathbf{B}_{2}:=\operatorname{diag}\left(\left[-k_{3}, k_{3}+k_{4},-k_{4}\right]\right) \\
\mathbf{B}_{3}:=\operatorname{diag}\left(\left[k_{3}, k_{3}-k_{4}, k_{4}\right]\right)
\end{array}\right.
$$

One deduces from Eqs. (27) and (29) the combined error dynamic linearization in the primed coordinates as

$$
\left[\begin{array}{c}
\dot{\mathbf{x}}  \tag{31}\\
\dot{\mathbf{y}}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{A}_{i} & \tilde{\mathbf{R}}_{i}^{\star} \\
\mathbf{B}_{i} & \underline{\boldsymbol{\Omega}}_{\times}
\end{array}\right]\left[\begin{array}{l}
\mathbf{x} \\
\mathbf{y}
\end{array}\right], \quad \text { with } i=0, \cdots, 3
$$

In order to prove the local exponential stability of the equilibrium $(\underline{\tilde{\mathbf{R}}}, \overline{\mathbf{b}})=\left(\mathbf{I}_{3}, \mathbf{0}\right)$, it suffices to prove that the origin of linear time-varying system (LTV) (31), with $i=0$, is uniformly exponentially stable. The proof is based on the results derived in [19, Theorem 1] which establish sufficient conditions for exponential stability of the LTV system having the form

$$
\left[\begin{array}{c}
\dot{\mathbf{x}}  \tag{32}\\
\dot{\mathbf{y}}
\end{array}\right]=\left[\begin{array}{cc}
\mathcal{A}(t) & \mathfrak{B}(t)^{\top} \\
-\mathcal{C}(t) & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\mathbf{x} \\
\mathbf{y}
\end{array}\right]
$$

By setting $\overline{\mathbf{y}}:=\underline{\mathbf{R}}^{\top} \mathbf{y}$, one easily verifies that System (31), with $i=0$, can be rewritten in the standard form (32) as

$$
\left[\begin{array}{c}
\dot{\mathbf{x}}  \tag{33}\\
\dot{\overline{\mathbf{y}}}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{A}_{0} & \underline{\mathbf{R}} \\
\underline{\mathbf{R}}^{\top} \mathbf{B}_{0} & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\mathbf{x} \\
\overline{\mathbf{y}}
\end{array}\right],
$$

with $\mathcal{A}(t)=\mathbf{A}_{0}, \mathfrak{B}(t)=\underline{\mathbf{R}}^{\top}, \mathcal{C}(t)=-\underline{\mathbf{R}}^{\top} \mathbf{B}_{0}$. Now we verify the two assumptions of Theorem 1 in [19]. First, the first assumption of this theorem is satisfied since $|\mathfrak{B}|$ and $\left|\frac{\partial \mathfrak{B}}{\partial t}\right|$ remain bounded for all time. Finally, the last assumption of this theorem is also satisfied since the symmetric matrices $\mathcal{P}=-\mathbf{B}_{0}$ and $\mathcal{Q}=2 \mathbf{A}_{0} \mathbf{B}_{0}$ satisfying the required relations $\mathcal{P} \mathfrak{B}^{\top}=\mathcal{C}^{\top}$ and $-\mathcal{Q}=\mathcal{A}^{\top} \mathcal{P}+\mathcal{P} \mathcal{A}+\dot{\mathcal{P}}$ are constant and positive definite. It remains to prove that $\mathfrak{B}$ is uniformly persistently exciting. This is also satisfied since for any positive numbers $\mu$ and $T>\mu$ one has

$$
\int_{t}^{t+T} \mathfrak{B}(\tau) \mathfrak{B}(\tau)^{\top} d \tau=\int_{t}^{t+T} \underline{\mathbf{R}}(\tau)^{\top} \underline{\mathbf{R}}(\tau) d \tau=T \mathbf{I}_{3}>\mu \mathbf{I}_{3}
$$

for all time $t \geq 0$. From here, the application of Theorem 1 in [19] ensures the uniform exponential stability of the origin of System (33) which in turn concludes the proof of Part 2 of the theorem statement.

In order to prove the instability of the three equilibria $(\underline{\tilde{\mathbf{R}}}, \overline{\mathbf{b}})=\left(\underline{\tilde{\mathbf{R}}}_{i}^{\star}, \mathbf{0}\right)$, with $i=1,2,3$, it suffices to show that the origin of the linearized system (31), with $i \in\{1,2,3\}$, is unstable. The proof is based on Chetaev's Theorem (see [16, Theorem 4.3]). Consider the following continuously differentiable functions:

$$
\mathcal{S}_{i}(\mathbf{x}, \mathbf{y}):=\frac{1}{2} \mathbf{x}^{\top} \mathbf{B}_{i} \underline{\tilde{\mathbf{R}}}_{i}^{\star} \mathbf{x}-\frac{1}{2}|\mathbf{y}|^{2}, \quad \text { with } i=1,2,3
$$

which are null at the origin, i.e. $\mathcal{S}_{i}(\mathbf{0}, \mathbf{0})=0$. Note that for each index $i \in\{1,2,3\}$, the matrix $\mathbf{B}_{i} \underline{\tilde{\mathbf{R}}}_{i}^{\star}$ has at least one element of the diagonal positive. For each index $i \in\{1,2,3\}$ and a number $r>0$, define

$$
U_{i, r}:=\left\{\xi:=(\mathbf{x}, \mathbf{y})^{\top}\left|\mathcal{S}_{i}(\xi)>0,|\xi|<r\right\}\right.
$$

and note that $U_{i, r}$ is non-null for all $r$. Let us now proceed the demonstration for index $i=1$. One verifies that

$$
\mathcal{S}_{1}(\xi)=k_{3} x_{1}^{2}+\left(k_{3}-k_{4}\right) x_{2}^{2}-k_{4} x_{3}^{2}-0.5|\mathbf{y}|^{2}
$$

This indicates that for all $\xi \in U_{1, r}$ one has

$$
k_{3} x_{1}^{2}+\left(k_{3}-k_{4}\right) x_{2}^{2}>k_{4} x_{3}^{2}+0.5|\mathbf{y}|^{2} \geq 0
$$

Using the condition $k_{3}>k_{4}$ in (13) it is straightforward to verify that $\mathbf{B}_{1} \underline{\tilde{\mathbf{R}}}_{1}^{\star} \mathbf{A}_{1}>0$ and, thus, $\dot{\mathcal{S}}_{1}(\xi)$ is positive for all $\xi \in U_{1, r}:$

$$
\begin{aligned}
\dot{\mathcal{S}}_{1}(\xi) & =\mathbf{x}^{\top} \mathbf{B}_{1} \tilde{\mathbf{R}}_{1}^{\star} \mathbf{A}_{1} \mathbf{x} \\
& =k_{1}\left(k_{3} x_{1}^{2}+\left(k_{3}-k_{4}\right) x_{2}^{2}\right)+k_{2} k_{4} x_{3}^{2} \\
& \geq k_{1}\left(k_{3} x_{1}^{2}+\left(k_{3}-k_{4}\right) x_{2}^{2}\right)>0 .
\end{aligned}
$$

It is seen that all the conditions of Chetaev's Theorem are satisfied. Therefore, the origin of System (31), with $i=1$, is unstable. Finally, the proofs of instability of the origin of System (31), with index $i \in\{2,3\}$, proceed analogously.

Some remarks are in order. At first glance the convergence and stability properties of the proposed observer (i.e. Properties 1, 2 and 3 of Theorem 1) are reminiscent of those of the standard observer (see [21, Theorem 5.1]). However, some novelties can be identified. The differences of the proposed observer (12) with respect to the standard one (3)-(4) lie in the definition of the innovation term $\sigma_{\mathbf{R}}$ and the dynamics of $\hat{\mathbf{b}}$. They allow for two intriguing properties, i.e. Properties 4 and 5 of Theorem 1:

- Since $\hat{\mathbf{a}}_{\mathcal{B}}=\hat{\mathbf{R}}^{\top} \mathbf{e}_{3}$ corresponds to the estimated roll and pitch angles, Property 4 of Theorem 1 implies the global decoupling of the dynamics of these estimated angles from the estimated yaw dynamics and from magnetometer measurements. This property has strong assets in practice as discussed in Subsection II-C, since obtaining a good estimation of roll and pitch independently from magnetic disturbances is especially important for robust flight stability of aerial robotic vehicles.
- Compared to the pure integrator $\hat{\mathbf{b}}=-k_{I} \int \sigma_{\mathbf{R}}$ in (3) which can grow arbitrarily large leading to integral windup effects as discussed in Subsection II-D, the proposed dynamics of $\hat{\mathbf{b}}$ given in (12) has certain assets. First, we have a pure integrator $\dot{\hat{\mathbf{b}}}=\sigma_{\mathbf{b}}$ as long as $|\hat{\mathbf{b}}(t)| \leq \Delta$, which allows for the compensation of the unknown constant gyro-bias b. Besides, as proved in Theorem 1, $|\hat{\mathbf{b}}(t)|$ is always bounded by the design threshold $\bar{\Delta}$. Finally, the gain $k_{b}$ can be chosen for the rate of desaturation of $\hat{\mathbf{b}}$ which can be seen, for instance, when $|\hat{\mathbf{b}}|$ is larger than $\Delta$ and $\sigma_{\mathbf{b}}=0$. The larger the value of $k_{b}$ the faster the rate of desaturation. Therefore, these properties allow one to effectively design the observer with limited integral wind-up effects.
An advantage of the proposed conditioned observer with respect to the standard observer, especially when the gravity and the geomagnetic vectors are close to each other, concerns the possibility of providing the system with fast dynamics without the use of high gains. Similarly to Subsection II-C, only for discussion purposes if one neglects the gyro-bias $\mathbf{b}$
and the dynamics of the estimated bias $\hat{\mathbf{b}}$, then the linearized error dynamics (31) can be reduced to $\dot{\mathbf{x}}=\mathbf{A}_{0} \mathbf{x}$, with $\mathbf{A}_{0}$ defined in (28). The poles of the roll/pitch and yaw error dynamics are independently given by $\lambda_{1}=\lambda_{2}=-k_{1}$ and $\lambda_{3}=-k_{2}$. Thus, non-high gains $k_{1}$ and $k_{2}$ can be chosen to provide the system with acceptably fast dynamics. Let us, as in Subsection (II-C), consider again the example of illconditioning of the two vectors $\mathbf{e}_{3}$ and $\overline{\mathbf{m}}_{\mathcal{I}}$ with $\bar{m}_{2} \approx 0$ and $\bar{m}_{3}^{2} \gg \bar{m}_{1}^{2}$. Since the three poles $\lambda_{1}^{s}, \lambda_{2}^{s}$ and $\lambda_{3}^{s}$, given in (7), of the linearized error dynamics of the standard observer are distinguished, it is impossible to choose the gains $k_{1}^{s}$ and $k_{2}^{s}$ involved in the standard observer so as to locally obtain similar error dynamics with the conditioned observer. In turn, it is possible to obtain the same dynamics in an axis corresponding to the roll, pitch or yaw dynamics. For example, if one would like to have similar dynamics in pitch with the same gain ratio $k_{1}^{s} / k_{2}^{s}=k_{1} / k_{2}=\kappa$, it suffices to choose $k_{i}^{s}=k_{i} \kappa /(1+\kappa)$, with $i=1,2$, so that $\lambda_{1}^{s}=k_{1}^{s}+k_{2}^{s}=\lambda_{1}=k_{1}$. In such a case, the third pole $\lambda_{3}^{s}$ is very close to zero if the gains $k_{1}^{s}$ and $k_{2}^{s}$ are not sufficiently high (see Eq. (7)), leading to excessively slow dynamics of the standard observer. On the other hand, if one would like the two observers to have similar dynamics in yaw, one should choose $k_{1,2}^{s}=k_{1,2}(1+\kappa) /\left(\kappa \bar{m}_{1}^{2}\right)$ so that

$$
\lambda_{3}^{s} \approx-\frac{k_{1}^{s} k_{2}^{s} \bar{m}_{1}^{2}}{k_{1}^{s}+k_{2}^{s}}=\lambda_{3}=-k_{2}
$$

Thus, in the case where $\bar{m}_{1}^{2} \ll 1$, the gains $k_{1}^{s}$ and $k_{2}^{s}$ become much higher than the gains $k_{1}$ and $k_{2}$ and excessively amplify the effects of measurement noises in the attitude estimate.

## IV. Practical Implementation Aspects

## A. Quaternion and Discrete Version

It is computationally expensive to compute the proposed filter (12) on the matrix representation of the attitude group $\mathrm{SO}(3)$ since the rotation matrix has 9 variables and 6 constraints. Instead, the use of the unit quaternion has three main advantages:

- The unit quaternion has no singularities (unlike Euler angle representations).
- It has four variables and a single constraint. Moreover, the constraint is a simple scalar renormalization.
- The transformation of a unit quaternion to a rotation matrix can be easily computed using Rodrigues' rotation formula [33], [35].
Since we begin with an algorithm $\mathrm{SO}(3)$ and lift it to an algorithm on the unit quaternions, there is no difficulties with the quaternion representation as discussed in [21], [23].

Let $\hat{\mathbf{q}} \in \mathbb{Q}$ be a unit quaternion associated with $\hat{\mathbf{R}}$. The proposed observer (12) can be rewritten as

$$
\left\{\begin{array}{l}
\dot{\hat{\mathbf{q}}}=\frac{1}{2} \mathbf{A}(\hat{\boldsymbol{\Omega}}) \hat{\mathbf{q}}, \quad \mathbf{q}(0) \in \mathbb{Q}  \tag{34}\\
\dot{\hat{\mathbf{b}}}=-k_{b} \hat{\mathbf{b}}+k_{b} \operatorname{sat}_{\Delta}(\hat{\mathbf{b}})+\sigma_{\mathbf{b}}, \quad|\hat{\mathbf{b}}(0)|<\Delta \\
\hat{\boldsymbol{\Omega}}:=\boldsymbol{\Omega}_{y}-\hat{\mathbf{b}}+\sigma_{\mathbf{R}}, \quad \mathbf{A}(\hat{\boldsymbol{\Omega}}):=\left[\begin{array}{cc}
0 & -\hat{\boldsymbol{\Omega}}^{\top} \\
\hat{\boldsymbol{\Omega}} & -\hat{\boldsymbol{\Omega}}_{\times}
\end{array}\right]
\end{array}\right.
$$

We proceed to derive a discrete version of observer (34). Assume that the sample time $T$ is small enough, so that the
approximation that $\hat{\boldsymbol{\Omega}}(t)$ and $\sigma_{\mathbf{b}}(t)$ remain constant over every period of time $S_{k}:=[k T,(k+1) T], \forall k \in \mathbb{N}$, is acceptable. Under this approximation, let us denote the value of $\hat{\boldsymbol{\Omega}}(t)$ and $\sigma_{\mathbf{b}}(t)$ over the period $S_{k}$ as $\hat{\boldsymbol{\Omega}}_{k}$ and $\sigma_{\mathbf{b}, k}$, respectively. Then, by exact integration of the first equation of (34) one obtains

$$
\hat{\mathbf{q}}_{k+1}=\exp \left(\frac{T}{2} \mathbf{A}\left(\hat{\boldsymbol{\Omega}}_{k}\right)\right) \hat{\mathbf{q}}_{k}
$$

Using the fact that $\mathbf{A}\left(\hat{\boldsymbol{\Omega}}_{k}\right)^{2}=-\left|\hat{\boldsymbol{\Omega}}_{k}\right|^{2} \mathbf{I}_{4}$, with $\mathbf{I}_{4}$ the identity matrix of $\mathbb{R}^{4 \times 4}$, one can verify from Taylor's expansions that $\exp \left(\frac{T}{2} \mathbf{A}\left(\hat{\boldsymbol{\Omega}}_{k}\right)\right)=\cos \left(\frac{T}{2}\left|\hat{\boldsymbol{\Omega}}_{k}\right|\right) \mathbf{I}_{4}+\frac{T}{2} \operatorname{sinc}\left(\frac{T}{2}\left|\hat{\boldsymbol{\Omega}}_{k}\right|\right) \mathbf{A}\left(\hat{\boldsymbol{\Omega}}_{k}\right)$, with $\operatorname{sinc}(\mathrm{s}):=\sin (s) / s, \forall s \in \mathbb{R}$. Consequently, the following discrete version of observer (34) is proposed

$$
\left\{\begin{array}{l}
\hat{\mathbf{q}}_{k+1}=\left(\cos \left(\frac{T \mid \hat{\boldsymbol{\Omega}}_{k \mid}}{2}\right) \mathbf{I}_{4}+\frac{T}{2} \operatorname{sinc}\left(\frac{T\left|\hat{\boldsymbol{\Omega}}_{k}\right|}{2}\right) \mathbf{A}\left(\hat{\boldsymbol{\Omega}}_{k}\right)\right) \hat{\mathbf{q}}_{k}  \tag{35}\\
\hat{\mathbf{b}}_{k+1}=T\left(-k_{b} \hat{\mathbf{b}}_{k}+k_{b} \operatorname{sat}_{\Delta}\left(\hat{\mathbf{b}}_{k}\right)+\sigma_{\mathbf{b}, k}\right)+\hat{\mathbf{b}}_{k}
\end{array}\right.
$$

In practice, for computational efficiency, the functions $\cos \left(\frac{T}{2}\left|\hat{\boldsymbol{\Omega}}_{k}\right|\right)$ and $\operatorname{sinc}\left(\frac{T}{2}\left|\hat{\boldsymbol{\Omega}}_{k}\right|\right)$ involved in (35) can be approximated by their first- or second-order approximation or by using a lookup table. Then, the estimated quaternion has to be renormalized since its unit norm constraint is generally not preserved.

## B. Fixed-point Format Implementation

Many microprocessors do not have a floating point unit (FPU) integrated. This means that any floating point calculation has to be emulated in a software leading to many fixedpoint calculations needed for a simple operation. The fixedpoint arithmetic can thus reduce the computational burden.

The attitude estimation algorithm of the present paper has been implemented on an 8-bit microcontroller (AVR Atmel ATMEGA644P) with 64 kbytes of Flash memory and running at 20 MHz . The code has been developed with the WinAVR development suite and implemented in fixed-point arithmetic, which means that every real number $r$ is transformed into an integer number $i$ by the transformation $i=$ floor $\left(r \times 2^{n}\right)$, where $n \in \mathbb{N}$ is the order of fixed-point format associated with $i$. For example, in fixed-point 14 format, i.e. $n=14$, if $r=0.314$ then $i=\operatorname{floor}\left(0.314 \times 2^{14}\right)=5144$. All the transformations from real to integer numbers are done during the code implementation, and when the code is executed, all of the calculations are done in fixed-point arithmetic only. In this way, very high execution rate of the estimation filter is achievable (for example, up to 500 Hz with the ATMEGA644P microcontroller including other processes like the flight control algorithm and data logging).

Two interesting issues having attracted our attention during earlier stages of code development deserve discussion: numerical overflow and underflow issues.

- Numerical overflow: We have opted to define the type of the estimated quaternion $\hat{\mathbf{q}}$ as "signed 16-bits integer" instead of "signed 32-bits integer" for computational efficiency reason. Let $n_{\mathbf{q}} \in \mathbb{N}$ denote the order of fixedpoint format associated with $\hat{\mathbf{q}}$. Then, it is of interest to have $n_{\mathbf{q}}$ as large as possible for maximizing the
precision of $\hat{\mathbf{q}}$. Since the first bit of a signed 16-bits integer is reserved for the sign bit and since theoretically each component of $\hat{\mathbf{q}}$ is bounded by 1 , one may choose $n_{\mathbf{q}}=15$. However, the numerical update of $\hat{\mathbf{q}}$ given in (35) may yield a component $\hat{q}_{i}$ of $\hat{\mathbf{q}}$ slightly larger than 1. In this case, one has $\hat{q}_{i} 2^{n_{\mathbf{q}}}>2^{15}$ and the overflow issue occurs. As a consequence, we choose $n_{\mathbf{q}}=14$ and proceed a renormalization of $\hat{\mathbf{q}}$ after each update according to (35).
- Numerical underflow: Let $n_{\mathbf{b}} \in \mathbb{N}$ be the order of fixedpoint format associated with the estimated gyro-bias $\hat{\mathbf{b}}$. Then, in view of the second equation of (35), a small value of $\sigma_{\mathbf{b}}$ can affect a change in the update of $\hat{\mathbf{b}}$ only if the term $\sigma_{\mathbf{b}} T$ is "visible" in the fixed-point $n_{\mathbf{b}}$ format, i.e. $\left|\sigma_{\mathbf{b}, i}\right| T 2^{n_{\mathbf{b}}} \geq 1$. Equivalently, in view of the definition of $\sigma_{\mathbf{b}}$ in (12), the terms $k_{3} \mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}$ should be "visible" in the fixed-point $n_{\mathbf{b}}$ format. Therefore, if $n_{\mathbf{b}}$ is not chosen sufficiently high, then the numerical underflow issue may occur. For example, if the estimation algorithm is run at 500 Hz (i.e. $T \approx 2^{-9}(\mathrm{~s})$ ), the gain $k_{3}$ is chosen small ( $k_{3} \approx 2^{-4}$ ), and $n_{\mathbf{b}}=14$, then the "visibility" of $k_{3} \mathbf{u}_{\mathcal{B}} \times$ $\hat{\mathbf{u}}_{\mathcal{B}}$ requires $k_{3} T 2^{n_{\mathbf{b}}}\left|\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)_{i}\right| \approx 2\left|\left(\mathbf{u}_{\mathcal{B}} \times \hat{\mathbf{u}}_{\mathcal{B}}\right)_{i}\right| \geq 1$, which roughly corresponds to an angle error greater than $\pi / 6$ (rad). Therefore, in view of the above discussion we have opted to stock the estimated variable $\hat{\mathbf{b}}$ in a "signed 32 -bits integer" format and choose $n_{\mathbf{b}}=28$ in order to avoid the numerical underflow issue.
The developed code has been successfully tested on a real UAV and the experimental results are reported in Section VI.


## C. Gain Tuning

The strategy of determining the gains for the proposed observer is inspired by the complementary filtering discussed in [21, App. A]. In fact, complementary filter provide an effective means of fusing multiple noisy measurements of the same signal that have complementary spectral characteristics [2], [21]. In our case, the attitude measurement provided by integration of the gyrometer measurements is predominantly disturbed by a low frequency noise (and subjected to a drift due to a slowly time-varying bias) while the accelerometer noise is a high frequency disturbance. Note also that the magnetometer measurements are disturbed by both low and high frequency noises, and that the strategy here proposed decouples the effects of the low frequency magnetometer noise from those of the accelerometer.

For convenience, similarly to the proof of Theorem 1 we consider the new attitude state and estimate $\underline{\mathbf{R}}=\mathbf{R}_{\alpha}^{\top} \mathbf{R}, \underline{\hat{\mathbf{R}}}=$ $\mathbf{R}_{\alpha}^{\top} \hat{\mathbf{R}}$, with $\mathbf{R}_{\alpha}$ defined in (18). We proceed by considering the linearization of the observer around the equilibrium $(\underline{\tilde{\mathbf{R}}}, \tilde{\mathbf{b}})=$ $\left(\mathbf{I}_{3}, \mathbf{0}\right)$ for the case where $\ddot{\mathbf{x}} \approx 0, \boldsymbol{\Omega} \approx \mathbf{0}$ and $\mathbf{R} \approx \mathbf{I}_{3}$. The sensor measurements satisfy

$$
\begin{aligned}
\mathbf{\Omega}_{y} & =\boldsymbol{\Omega}+\eta_{\Omega}+\mathbf{b} \\
\mathbf{a}_{\mathcal{B}} & \approx-g \mathbf{R}^{\top} \mathbf{e}_{3}+\eta_{\mathbf{a}}, \\
\mathbf{m}_{\mathcal{B}} & =\mathbf{R}^{\top} \mathbf{m}_{\mathcal{I}}+\eta_{\mathbf{m}},
\end{aligned}
$$

where $\eta_{(\cdot)}$ represents noise in the measurements and $\mathbf{b}$ is a deterministic perturbation dominated by low-frequency content.

Thus, one deduces

$$
\begin{equation*}
\mathbf{u}_{\mathcal{B}}=\underline{\mathbf{R}}^{\top} \mathbf{e}_{3}+\eta_{\mathbf{u}}, \quad \mathbf{v}_{\mathcal{B}}=\underline{\mathbf{R}}^{\top} \mathbf{e}_{1}+\eta_{\mathbf{v}}, \tag{36}
\end{equation*}
$$

with

$$
\eta_{\mathbf{u}}=-\frac{\eta_{\mathbf{a}}}{g}, \eta_{\mathbf{v}} \approx-\frac{\left(\eta_{\mathbf{u}} \mathbf{e}_{3}^{\top}+\mathbf{e}_{3} \eta_{\mathbf{u}}^{\top}\right) \mathbf{m}_{\mathcal{I}}+\pi_{\mathbf{e}_{3}} \eta_{\mathbf{m}}}{\left|\pi_{\mathbf{e}_{3}} \mathbf{m}_{\mathcal{I}}\right|}
$$

Proceed the linearization computations analogously as in Eqs. (25), (26), (27), (29) with $\mathbf{u}_{\mathcal{B}}$ and $\mathbf{v}_{\mathcal{B}}$ given by (36) one can verify that in a first order approximation $\underline{\tilde{\mathbf{R}}}=\mathbf{I}_{3}+\mathbf{x}_{\times}$and

$$
\left\{\begin{array}{l}
\dot{\mathbf{x}}=\mathbf{A}_{0} \mathbf{x}-\eta_{\boldsymbol{\Omega}}-\mathbf{b}+\hat{\mathbf{b}}+k_{1} \mathbf{e}_{3 \times} \eta_{\mathbf{u}}+k_{2} \mathbf{e}_{3} \mathbf{e}_{2}^{\top} \eta_{\mathbf{v}}  \tag{37}\\
\dot{\hat{\mathbf{b}}}=\mathbf{B}_{0} \mathbf{x}-k_{3} \mathbf{e}_{3 \times} \eta_{\mathbf{u}}-k_{4} \mathbf{e}_{1 \times} \eta_{\mathbf{v}}
\end{array}\right.
$$

with $\mathbf{A}_{0}$ and $\mathbf{B}_{0}$ defined in (28) and (30), respectively. Let $(\Phi, \Theta, \Psi)$ and ( $\hat{\Phi}, \hat{\Theta}, \hat{\Psi}$ ) denote the Euler angles associated with $\underline{\mathbf{R}}$ and $\underline{\hat{\mathbf{R}}}$, respectively. Using the approximations

$$
\begin{aligned}
& x_{1} \approx \Phi-\hat{\Phi}, x_{2} \approx \Theta-\hat{\Theta}, x_{3}=\Psi-\hat{\Psi} \\
& \dot{\Phi} \approx \Omega_{1}, \dot{\Theta} \approx \Omega_{2}, \dot{\Psi} \approx \Omega_{3} \\
& \Phi_{a c c}=u_{\mathcal{B}, 2} \approx \Phi+\eta_{\mathbf{u}, 2}, \quad \Theta_{a c c}=-u_{\mathcal{B}, 1} \approx \Theta-\eta_{\mathbf{u}, 1} \\
& \Theta_{m a g}=-v_{\mathcal{B}, 3} \approx \Theta-\eta_{\mathbf{v}, 3}, \quad \Psi_{m a g}=v_{\mathcal{B}, 2} \approx \Psi+\eta_{\mathbf{v}, 2}
\end{aligned}
$$

one deduces from (37) that

$$
\left\{\begin{array}{l}
\dot{\hat{\Phi}}=-k_{1} \hat{\Phi}+k_{1} \Phi_{a c c}+\Omega_{y, 1}-\hat{b}_{1} \\
\dot{\hat{\Theta}}=-k_{1} \hat{\Theta}+k_{1} \Theta_{a c c}+\Omega_{y, 2}-\hat{b}_{2} \\
\dot{\hat{\Psi}}=-k_{2} \hat{\Psi}+k_{2} \Psi_{m a g}+\Omega_{y, 3}-\hat{b}_{3} \\
\dot{\hat{b}}_{1}=-k_{3} \Phi_{a c c}+k_{3} \hat{\Phi} \\
\dot{\hat{b}}_{2}=-k_{3} \Theta_{a c c}-k_{4} \Theta_{m a g}+\left(k_{3}+k_{4}\right) \hat{\Theta} \\
\dot{\hat{b}}_{3}=-k_{4} \Psi_{m a g}+k_{4} \hat{\Psi}
\end{array}\right.
$$

In the Laplace domain with the Laplace variable $s$, one obtains

$$
\left\{\begin{array}{l}
\hat{\Phi}(s)=T_{\Phi}(s) \Phi_{a c c}+S_{\Phi}(s) \frac{\Omega_{y, 1}}{s} \\
\hat{\Theta}(s)=T_{\Theta}^{a}(s) \Theta_{a c c}+T_{\Theta}^{m}(s) \Theta_{m a g}+S_{\Theta}(s) \frac{\Omega_{y, 2}}{s} \\
\hat{\Psi}(s)=T_{\Psi}(s) \Phi_{m a g}+S_{\Psi}(s) \frac{\Omega_{y, 3}}{s}
\end{array}\right.
$$

with the complementary transfer functions satisfying $\sum_{i} T_{(\cdot)}^{i}(s)+S_{(\cdot)}(s)=1:$

$$
\left\{\begin{array}{l}
T_{\Phi}(s)=\frac{k_{1} s+k_{3}}{s^{2}+k_{1} s+k_{3}}, \quad S_{\Phi}(s)=\frac{s^{2}}{s^{2}+k_{1} s+k_{3}} \\
T_{\Theta}^{a}(s)=\frac{k_{1} s+k_{3}}{s^{2}+k_{1} s+k_{3}+k_{4}}, T_{\Theta}^{m}(s)=\frac{k_{4}}{s^{2}+k_{1} s+k_{3}+k_{4}} \\
S_{\Theta}(s)=\frac{s^{2}}{s^{2}+k_{1} s+k_{3}+k_{4}}, \\
T_{\Psi}(s)=\frac{k_{2} s+k_{4}}{s^{2}+k_{2} s+k_{4}}, \quad S_{\Psi}(s)=\frac{s^{2}}{s^{2}+k_{2} s+k_{4}}
\end{array}\right.
$$

In the case of proportional feedback, i.e. $k_{3}=k_{4}=0$, the crossover frequency of the complementary filters for the roll/pitch- and yaw axes are respectively given by the proportional (P-) gains $k_{1}$ and $k_{2}$. The P-gain $k_{1}$ (respectively, $k_{2}$ ) is typically chosen for the best crossover frequency in order to trade-off between a low-pass filter of the accelerometer
(respectively, magnetometer) measurements and a high-pass filter of the attitude measurement obtained by integration of the gyrometer measurements.

The integral (I-) gains $k_{3}$ and $k_{4}$ govern the dynamics of the gyro-bias estimates. Since the dynamics of the real biases are slowly time-varying compared to those of the roll, pitch and yaw angles, the I-gains $k_{3}$ and $k_{4}$ should be chosen at least 10 times smaller than $k_{1}$ and $k_{2}$, respectively, in order to ensure a good time-scale separation between the estimation dynamics of the angles and the gyro biases. In practice, we choose the P-gain $k_{1}$ approximately equal to 1 and the I-gain $k_{3}$ at 16 or 32 times smaller than $k_{1}$. The division by 16 or 32 is adopted because in a fixed-point implementation it can be easily done by bit shifts. Figure 2 shows the Bode plots of the sensitivity $S_{\Phi}(s)$ and complementary sensitivity $T_{\Phi}(s)$ for different sets of gains. In the case where the I-gain $k_{3}$ is not sufficiently smaller than $k_{1}$ (for example, $k_{3}=k_{1} / 3$ ), the Bode plots of $S_{\Phi}(s)$ and $T_{\Phi}(s)$ are no longer reminiscent of those of a first order system, whereas in the case $k_{3}=k_{1} / 32$ similarity of the Bode plots is clearly visible.

Since for aerial robotic applications the measurement of the geomagnetic field is less reliable than that of the gravity direction, we choose the gain $k_{2}$ about 5 times smaller the $k_{1}$. This also means that the complementary filter relies more on the gyrometers to track the yaw dynamics. The ratio $k_{1} / k_{3}$ and $k_{2} / k_{4}$ is chosen equal so that the time-scale separation between angle and gyro-bias estimation is the same for roll and yaw axes. As a consequence, the gain $k_{4}$ becomes very small. In this case, the sensitivity function $T_{\Theta}^{m}(s)$ is negligible, and good approximations $S_{\Theta}(s) \approx S_{\Phi}(s)$ and $T_{\Theta}^{a}(s) \approx T_{\Phi}(s)$ can be obtained.


Fig. 2. Bode plots of $S_{\Phi}(s)$ and $T_{\Phi}(s)$ for the sets $\left\{k_{1}=1, k_{3}=0\right\}$, $\left\{k_{1}=1, k_{3}=k_{1} / 32\right\}$ and $\left\{k_{1}=1, k_{3}=k_{1} / 3\right\}$.

## V. Simulation Results

In this section we illustrate through simulation results the improved performance of the conditioned observer (12)
compared to the standard implementation of the explicit complementary filter proposed in [10], [21].

Simulations are carried on for the following scenario: an IMU is fixed to a vertical take-off and landing (VTOL) vehicle which is in stationary flight so that its attitude matrix $\mathbf{R}$ is $\mathbf{I}_{3}$, i.e. $\phi=\theta=\psi=0$. The normalized geomagnetic field expressed in the inertial frame $\mathcal{I}$ is taken as $\mathbf{m}_{\mathcal{I}}=$ $(0.4334,0.0012,0.9012)^{\top}$.

The gains and parameters involved in the conditioned observer (12) are given by

$$
\begin{equation*}
k_{1}=1, k_{2}=0.2, k_{3}=\frac{k_{1}}{32}, k_{4}=\frac{k_{2}}{32}, k_{b}=16, \Delta=0.03 \tag{38}
\end{equation*}
$$

The P-gain $k_{1}$ is chosen larger than $k_{2}$ since we assume that the measurement of the gravity direction is more reliable than that of the geomagnetic field. The I-gains $k_{3}$ and $k_{4}$ are chosen small compared to the P-gains $k_{1}$ and $k_{2}$ in order to avoid coupling of attitude and bias dynamics and reduce integral wind-up effects. The value of $\Delta$ corresponds to an estimated bound of each component of the gyro-bias vector $\mathbf{b}$ equal to $1 \mathrm{deg} / \mathrm{s}$. The gain $k_{b}$ is chosen large in order to obtain a fast desaturation rate of $\hat{\mathbf{b}}$.

As discussed in Section III, it is impossible to choose a set of gains for the standard observer so as to provide similar dynamics in roll, pitch and yaw with those of the conditioned observer. However, it is possible to choose these gains so as to obtain similar dynamics in roll, pitch, or yaw with the corresponding one of the conditioned observer. Two simulations are reported.

Simulation 1: The gains involved in the standard observer are chosen as

$$
k_{1}^{s}=1, k_{2}^{s}=0.2, k_{I}=\frac{1}{32}
$$

allowing the linearized error system to have similar dynamics in pitch with that of the conditioned observer whose gains and parameters are given in (38). No noise in gyrometerand accelerometer measurements is introduced. By contrast, a constant gyro-bias vector $\mathbf{b}=(0.01,-0.005,-0.01)^{\top}(\mathrm{rad} / \mathrm{s})$ is added. Besides, each component of the magnetometer measurement vector $\mathbf{m}_{\mathcal{B}}$ is corrupted by an additive white Gaussian noise of variance 0.3 -a very large value in view of the norm equal to 1 of the vector $\mathbf{m}_{\mathcal{I}}$. The initial estimated Euler angles associated with the initial estimated attitude matrix $\mathbf{R}(0)$ are rather large, i.e. $(\hat{\phi}(0), \hat{\theta}(0), \hat{\psi}(0))=$ $(-45,45,90)(\mathrm{deg})$, and the initial estimated gyro-bias is taken as $\hat{\mathbf{b}}(0)=(0,0,0)^{\top}(\mathrm{rad} / \mathrm{s})$. The results illustrated in Figures 3 and 4 show important performance differences between the proposed conditioned observer and the standard observer. In particular, the latter would yield important overshoots and oscillations in both the estimated Euler angles and the estimated gyro-bias components, and also a very slow convergence in the yaw estimate. In contrast, one can observe, for the conditioned observer, a very fast convergence of the estimated variables to the real values, and the quasi absence of overshoots of the estimated attitude despite the use of the integral correction term $\hat{\mathbf{b}}$ and the large initial estimation errors. It can also be seen that the magnetic disturbances do not degrade the estimation performance of roll and pitch estimates and of


Fig. 3. Estimated and real Euler angles (simulation 1).


Fig. 4. Gyro-bias error $\tilde{\mathbf{b}}=\mathbf{b}-\hat{\mathbf{b}}$ (simulation 1 ).
the first and second components of the gyro-bias estimate $\hat{\mathbf{b}}$, whereas the corresponding estimated variables of the standard observer are effected. On the other hand, a slightly higher amplification of magnetometer noises on the yaw estimate is a price to pay for faster dynamics compared to the standard observer.

Simulation 2: The gain involved in the standard observer are chosen as

$$
\left\{\begin{array}{l}
k_{1}^{s}=\frac{k_{1}\left(1+k_{1} / k_{2}\right)}{k_{1} / k_{2} m_{1}^{2}}=6.371, \\
k_{2}^{s}=\frac{k_{2}\left(1+k_{1} / k_{2}\right)}{k_{1} / k_{2} m_{1}^{2}}=1.274, k_{I}=\frac{1}{32}
\end{array}\right.
$$



Fig. 5. Estimated and real Euler angles (simulation 2).


Fig. 6. Gyro-bias error $\tilde{\mathbf{b}}=\mathbf{b}-\hat{\mathbf{b}}$ (simulation 2).
allowing the linearized error system (as discussed in Section III) to have similar dynamics in yaw with that of the conditioned observer. The same constant gyro-bias vector b, magnetometer measurement noises and initial estimated attitude and gyro-bias $(\hat{\mathbf{R}}(0), \hat{\mathbf{b}}(0))$ as in Simulation 1 are introduced. In addition, each component of the accelerometer measurement vector $\mathbf{a}_{\mathcal{B}}$ is corrupted by an additive white Gaussian noise of variance 1 . Figures 5 and 6 show clearly a better performance of the conditioned observer compared to that of the standard observer. Whilst similar convergence and rather smooth behavior in the yaw estimate of the two observers are obtained, the roll and pitch estimates and the first and second components of the gyro-bias estimate $\hat{\mathbf{b}}$
provided by the conditioned observer are much less noisy than that of the standard observer. Indeed, too high value of $k_{1}^{s}$ involved in the standard observer overly amplifies the effects of measurement noises in the estimated variables.

## VI. Experimental Results

The experiments were performed on a quadrotor helicopter which is equipped with a low-cost IMU composed of a 3-axis accelerometer (MXR9500) and three single-axis gyrometers (ADXRS610), and a magnetometer (HMC5883L magnetic sensor). The attitude estimated by the algorithm of this paper is compared with "ground truth" measurement data acquired by a motion capture system from Vicon. This vision-based system is composed of 8 cameras mounted on the ceiling of the flying room of the Autonomous Systems Lab (ASL) at the ETH Zurich, where the experiments took place. The Vicon system provides the full pose of the flying vehicle at a rate of 200 Hz .


Fig. 7. Euler angles $i$ ) estimated by $i . a)$ the proposed conditioned observer, i.b) simply integrating the kinematics equation of rotation using gyrometer measurement data, and ii) measured by the Vicon system

The proposed discrete version of the attitude estimator, i.e. observer (35), is implemented on the computer aboard the helicopter with a sampling period of 200 Hz . The gains and parameters involved in the proposed conditioned observer are given in Section V. The experimental results are reported in Figure 7. The estimated values for the pitch and roll angles are very close to the ground truth, meaning that the bias has been properly estimated and removed. Regarding the estimation of the yaw angle, the presence of a bias is visible. This may be
explained by the fact that the inertial magnetic field vector inside the flying room of the ASL is not exactly known and might be slightly perturbed by the electrical equipment in this area. The presence of magnetic disturbances does not prevent the estimates of the roll and pitch angles to converge to the true value, thus confirming experimentally the discussion about decoupling in Section III.

## VII. Conclusions

In this paper a novel nonlinear attitude observer is proposed, allowing for the global decoupling of the estimation of the roll and pitch angles from the estimation of the yaw angle and from the presence of magnetic disturbances. It also allows for the compensation of the gyrometers' bias of a low-cost IMU using an anti-windup integration technique. Practical implementation aspects such as discrete implementation in quaternions, gain tuning and fixed-point implementation are presented. Finally, simulation and experimental results have supported the approach.

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