# Visual Servoing for Underactuated VTOL UAVs : a Linear, Homography-Based Framework 

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#### Abstract

Résumé The paper concerns the control of Vertical Take Off and Landing (VTOL) Underactuated Aerial Vehicles (UAVs) in hover flight, based on measurements provided by an on-board video camera. The objective is to stabilize the vehicle to the equilibrium pose associated with an image of a planar target, using a minimal sensor suite and poor knowledge of the environment. By using the homography matrix computed from the camera measurements of the target, stabilizing feedback laws are derived based on the visual data and gyrometer measurements only. Explicit stability conditions on the control parameters are provided, showing that a proper tuning of the control parameters ensures a large robustness margin with only planar target and visibility assumptions, while the target size and orientation, the UAV position, linear velocity and orientation are unknown. Additional issues, such as the use of accelerometers in order to improve the UAV's positioning in the case of unmodeled dynamics (like wind), are also considered.


## 1 Introduction

Unmaned Aerial Vehicles (UAVs) can be used for many surveillance and monitoring applications, both indoor and outdoor environment. Their effectiveness relies in the first place on the use of embarked sensors that can provide information on the vehicule's pose (i.e. position and orientation). In teleoperated modes, the human operator can compensate for the lack of some pose information (like, e.g., the vehicule's position). For fully autonomous control modes, however, information on both position and orientation is necessary. It is well known that pose estimation is a challenging problem for UAVs, and especially for VTOLs (Vertical Take-Off and Landing vehicles). This is due to several reasons, among which, $i$ ) the absence of sensors that can provide a direct measure of the 3D-orientation, ii) the difficulty to obtain precise and high-rate position measurements via GPS sensors, iii) the impossibility to use these sensors in some environments (like, e.g., urban canyons). Thus, while interesting results have been obtained with Inertial Measurement Unit (IMU) [18], or GPS-aided IMU (see, e.g., [21, 13] for recent results), other sensors are needed to improve UAV's effectiveness, especially those providing information

[^0]about UAV's local environment. One of the most promising alternatives is vision sensors. Cameras provide a rich information about the environment. They have been used extensively for ground robotics applications. Over the last ten years, vision-based control solutions have been developed for aerial vehicles. Regulation of a mechanical system based on visual features for feedback is known as Visual Servo Control [9]. There are two main approaches in visual servo control [5, 6] : Image-Based Visual Servoing (IBVS), and Position-Based Visual Servoing (PBVS), depending upon whether the controller is designed to directly act on the visual information (IBVS) or whether the visual information is first used in the pose reconstruction (PBVS). The latter has been successfully implemented on a number of aerial vehicles [25, 26, 1, 23]. It requires, however, an accurate geometric model of the visual target along with good calibration of the camera. An IBVS scheme was first presented in [12] for underactuated systems. The dynamics of features in image space were formulated in terms of their spherical projections to preserve the dynamic structure of the system and used as direct inputs to the control algorithm. The controller was designed to stabilize the dynamics of the image features. Extensions of this work have been recently proposed and successfully implemented, especially to improve the conditioning of the Jacobian matrix [11] or to overcome the need for a velocity sensor, using the measurement of the optical flow [16]. These approaches do not require accurate geometric target models or well calibrated cameras, but they lead to complex nonlinear control problems due to the appearance of the image depth as an unknown scale factor into the system dynamics. A comparison of different IBVS control schemes for VTOL UAVs can be found in [4]. When the target is planar, an alternative approach is the Homography-Based Visual Servoing (HBVS), originally developed for robot manipulators or more general fully-actuated systems [20, 3]. HBVS uses the homography matrix as feedback input. This matrix relates the coordinates of the target's points in camera frame as seen from two configurations of the camera. It can be computed from two images of the same planar scene. Several efficient computer vision algorithms have been proposed to this purpose (see e.g. [3]). While the homography matrix can be expressed in term of geometric parameters : camera displacement, normal to the target plane, and distance from the target to the camera, its decomposition in term of these parameters is ill-posed due to the existence of multiple solutions. Consequently, prior works on HBVS of underactuated vehicles exploit additional information on the camera displacement from other sensors (e.g. orientation from IMU). Examples are given in [27] (and subsequently in [24], [7]) in a $2 \mathrm{D} \frac{1}{2}$ visual servoing formulation, in [26] for a landing manoeuver of helicopter on a fixed target, or more recently used for landing of fixed wing aircrafts $[15,10]$. In all these cases, the use of additional information on the pose makes HBVS strongly related to PBVS solutions. To the authors knowledge, there is no stabilisation results for underactuated systems relying on the homography matrix only.

The main contribution of this paper is to show that the stabilisation of VTOL UAVs in hover flight can be achieved based on measurements provided by a single camera and gyrometers only. Two control laws are proposed to address the case of a non-vertical and non-horizontal target respectively, with the natural assumption that the target is in the camera field of view at the equilibrium pose. Each of these controllers is thus applicable for a $\pm 90^{\circ}$ range of target orientation, knowing that the control expression does not make use of the value of the target's normal. The control approach proposed in this paper exploits directly the homography matrix that encodes the pose information of the camera with respect to the target. It is assumed, however, that this matrix is available for control computation, e.g. using computer vision algorithms mentionned above. It is also assumed that the target is visible and remains inside the camera's field. This is a
reasonable assumption since the stabilization discussed here is local. There are several challenges associated with this problem. First, since the target's orientation is not known, the vehicle's pose cannot be extracted from the homography measure. In addition, unlike many works on the subject, we do not assume that the vehicle's orientation can be reconstructed. In fact, such a reconstruction is difficult in practice and in particular, the use of accelerometers to recover the system's attitude remains controversial [22]. Then, we do not have any sensor that provides linear velocity measurements either. Finally, the systems here considered are underactuated (i.e. the number of independent force and torque controls is strictly smaller than the number of degrees of freedom). The approach builds on a previous result by Benhimane and Malis [3] for the control of robotic manipulators, based on a kinematic and holonomic model. The fact that dynamical models of underactuated vehicles are considered here makes the problem significantly harder. Note that a preliminary version of this paper was presented in [8]. With respect to that work, the present paper contains several extensions. First, the solution proposed in [8] did not address the case of a vertical target. That result is here extended in order to include this case. Then, the assumption according to which the camera's optical axis is aligned with the vehicle's thrust direction is relaxed : no assumption on the camera's orientation (except for the natural visibility assumption of the target) is needed anymore. We also investigate the use of accelerometer measurements to improve the UAV's positioning in the presence of unmodeled dynamics (like wind). Finally, proofs of the proposed results are provided.

The paper is organized as follows. Section 2 reviews some technical background and provides a precise description of the addressed problem. Section 3 contains the main results of the paper : stabilizing control laws computed from homography measurements are proposed, together with explicit stability conditions. These results are complemented in Section 4 in two ways. Firstly, the use of accelerometers to improve the positionning in the presence of unmodeled dynamics is addressed. Secondly, a gain-tuning strategy that allows to obtain good performance in a large operating domain is proposed. Simulation results validating the control approach are presented in Section 5 . The paper ends with concluding remarks.

## 2 Preliminary background

### 2.1 Problem statement

The problem addressed in this paper (see Fig. 1 and 2 below for illustration and notation) corresponds to a typical scenario for UAVs. The vehicle is equipped with a camera. A reference image of a planar target $\mathcal{T}$ is taken at some desired pose (i.e. location), represented by the reference frame $\Re^{*}$. Based on this reference image and the current image, the objective is to design feedback laws that stabilize the vehicle at the desired pose.

Except for the planarity assumption no other information on the target, like geometry or orientation, is available. In particular, the target's normal is unknown. The distance to the target at the desired pose is also unknown, although a (very rough) lowerbound on this distance is needed to guarantee stability. We shall assume, however, that the target remains in the field of view of the camera along the trajectory. This assumption is reasonable in the present case since the addressed stabilization problem (and stabilization result) is essentially local.

Since the systems here considered are underactuated, we have to assume that the desired pose is an equilibrium of the vehicle. Otherwise the problem of asymptotic stabilization cannot be solved. For example, in the case of an helicopter, neglecting the lateral force due to the anti-torque rotor,


Figure 1 - Problem scheme : stabilization w.r.t. a ground target


Figure 2 - Problem scheme : stabilization w.r.t. a frontal target
stabilization of a desired pose requires that the rotor thrust direction is vertical at this pose (in
the absence of wind), so as to compensate gravity without inducing lateral motion. This fixes two rotational degrees of freedom.

### 2.2 Dynamics of thrust-propelled underactuated vehicles

The proposed approach applies to the class of underactuated "thrust-propelled" VTOL vehicles [14]. More precisely, we consider rigid bodies with one force control in a body-fixed direction and full torque actuation. Typical examples are given by helicopters, ducted fans, quadrotors, etc. To comply with the assumption that the reference pose is an equilibrium for the vehicle, it is assumed that the thrust direction at the reference pose is aligned with the vertical basis vector of the reference frame $\Re^{*}$. The dynamical equations are then given by

$$
\left\{\begin{align*}
\dot{p} & =R v  \tag{1}\\
\dot{R} & =R S(\omega) \\
m \dot{v} & =-m S(\omega) v-T b_{3}+m \gamma \\
J \dot{\omega} & =-S(\omega) J \omega+\Gamma
\end{align*}\right.
$$

with $p$ the position vector of the vehicle's center of mass, expressed in the reference frame, $R$ the rotation matrix from the current (i.e. body) frame to the reference frame, $v$ the linear velocity vector with respect to (w.r.t.) the reference frame expressed in the current frame, $\omega$ the angular velocity vector expressed in the current frame, $S($.$) the matrix-valued function associated with$ the cross product, i.e. $S(x) y=x \times y, \forall x, y \in \mathbb{R}^{3}, m$ the mass, $T$ the thrust input, $b_{3}=(0,0,1)^{T}$, $J$ the inertia matrix, $\Gamma$ the torque vector, and $\gamma=g R^{T} b_{3}$ the projection of the gravity vector in the current frame with $g$ the gravity constant.

The first and second equations of System (1) correspond to the kinematics, while the third and fourth account for the dynamics. The objective is to asymptotically stabilize the origin $p=$ $0, R=I_{3}, v=0, \omega=0$, with $I_{3}$ the $3 \times 3$ identity matrix, from visual measurements of a reference image $\mathcal{I}^{*}$ (taken at $\Re^{*}$ ) and the current image $\mathcal{I}$ (taken at $\Re$ ) of the planar target $\mathcal{T}$. Note in particular that neither $p$ nor $R$ are directly measured. Visual measurements only provide a partial and coupled measurement of these quantities (see subsection below). The linear velocity vector $v$ is not measured either. On the other hand, we assume that the angular velocity $\omega$ is measured via gyrometers. We review below some well known facts about visual sensors and homography matrices.

### 2.3 Visual observation of planar scenes and homography matrices

Let us first assume that the camera and UAV frame coincide (this assumption is relaxed in the next section). The following notation applies to the planar scene $\mathcal{T}$ (see Fig. 1 and 2)

- $\chi^{*}, \chi$ are the coordinates of a point of interest $\mathcal{P}$ lying on the planar target, expressed in the reference and current frames respectively.
- $n^{*}$ is the unit vector defining the normal to the planar object, expressed in the reference frame ; $d^{*}$ is the distance between this plane and the camera optical center. $Z^{*}$ is the third coordinate of point $\mathcal{P}$ in the reference frame.
A useful tool in visual servoing is the so-called homography matrix $\mathbf{H}$ which embeds all information regarding the transformation between two images of the same planar object of interest (see, e.g., [3, 17] for more details). An important feature of this matrix is that it can be estimated
from these images without any assumption on the camera pose. $\mathbf{H}$ is defined as :

$$
\begin{equation*}
\mathbf{H} \triangleq K\left(R^{T}-\frac{1}{d^{*}} R^{T} p n^{* T}\right) K^{-1} \tag{2}
\end{equation*}
$$

with $K$ the camera intrinsic parameters matrix. The matrix $\mathbf{H}$ relates the normalized coordinates of a point as seen from the reference and current pose. Indeed, the following relationship holds :

$$
\begin{equation*}
\chi=R^{T} \chi^{*}-R^{T} p \tag{3}
\end{equation*}
$$

Defining the pixel coordinates - which embed the camera calibration error- $\mu=K\left(\frac{1}{Z} \chi\right)$ and $\mu^{*}=K\left(\frac{1}{Z^{*}} \chi^{*}\right)$ and noticing that $n^{* T} \chi^{*}=d^{*}$, one gets :

$$
\begin{align*}
\frac{Z}{Z^{*}} \mu & =K\left(R^{T} \frac{\chi^{*}}{Z^{*}}-R^{T} \frac{p}{Z^{*}} \frac{1}{d^{*}} n^{* T} \chi^{*}\right) \\
& =K\left(R^{T}-\frac{1}{d^{*}} R^{T} p n^{* T}\right) K^{-1} \mu^{*}=\mathbf{H} \mu^{*} \tag{4}
\end{align*}
$$

The scalar $\frac{Z}{Z^{*}}$ is the —unknown- ratio of the third coordinates in both frames. This relationship suggests that $\mathbf{H}$ can be estimated only up to an unknown scalar factor. Several algorithms have been proposed for the estimation of the Homography matrix (see, e.g., [3, 17]). Assuming that $K$ is known, one can compute an estimate of the matrix

$$
H_{\eta}=\eta\left(R^{T}-\frac{1}{d^{*}} R^{T} p n^{* T}\right)
$$

with $\eta$ some scalar factor. One can show (see, e.g., [17, Pg. 135]) that $\eta$ corresponds to the mean singular value of $H_{\eta}$. Furthermore, an explicit formula for the calculation of $\eta$ is proposed in [19, App. B]. Therefore, we assume from now on the knowledge of

$$
\begin{equation*}
H=R^{T}-\frac{1}{d^{*}} R^{T} p n^{* T} \tag{5}
\end{equation*}
$$

The time-derivative of $H$ is easily deduced from (1) :

$$
\dot{H}=-S(\omega) H-\frac{1}{d^{*}} v n^{* T}
$$

In [3], $H$ was used to define an error vector and an associated feedback law, based on a kinematic control model. More precisely, the following result was shown.

Proposition 2.1 [3, Sec. 4] Assume that the camera optical axis corresponds to the $z$-axis of the body frame. Let $\chi^{*}$ denote the coordinates of a point $\mathcal{P} \in \mathcal{T}$, expressed in the reference frame $\Re^{*}$, and $m^{*}=\frac{1}{Z^{*}} \chi^{*}$ the associated normalized coordinates. Let $e \in \mathbb{R}^{6}$ denote the error vector defined by

$$
\begin{equation*}
e=\binom{e_{p}}{e_{\Theta}}, \quad e_{p}=(I-H) m^{*}, \quad e_{\Theta}=\operatorname{vex}\left(H^{T}-H\right) \tag{6}
\end{equation*}
$$

with vex the inverse operator of any valued skew matrix : $\operatorname{vex}(S(x))=x$ for any $x \in \mathbb{R}^{3}$. Then,

1. $(p, R) \longmapsto e$ defines a local diffeormorphism around $(p, R)=\left(0, I_{3}\right)$. In particular, $e=0$ if and only if $(p, R)=\left(0, I_{3}\right)$.
2. The kinematic control law

$$
\begin{equation*}
v=-\lambda_{p} e_{p}, \quad \omega=-\lambda_{\Theta} e_{\Theta} \tag{7}
\end{equation*}
$$

with $\lambda_{p}, \lambda_{\Theta}>0$ makes $(p, R)=\left(0, I_{3}\right)$ locally asymptotically stable.
Remark 2.2 1) In [3], $e_{p}$ and $e_{\Theta}$ are defined with an opposite sign, i.e. $e_{p}=(H-I) m^{*}$, $e_{\Theta}=$ $\operatorname{vex}\left(H-H^{T}\right)$. The present choice is better adapted to the definition of $v$ and $\omega$ in (1). 2) There is no constraint on $m^{*}$ except that it must be a projective vector, i.e., $m_{3}^{*}=1$. Note, however, that there is an implicit constraint on the target's orientation, i.e. $n_{3}^{*}>0$, corresponding to the fact that the optical (semi)-axis of the camera intersects the planar target.

### 2.4 General camera configuration

When the camera and body frames do not coincide, the expression of the homography matrix in terms of $p$ and $R$ is different. Let us denote by $\chi_{c}^{*}\left(\right.$ resp. $\left.\chi_{c}\right)$ the coordinates of $P$ in the reference (resp. current) camera frame. One has

$$
\left\{\begin{align*}
\chi_{c} & =R_{c} \chi+p_{c}  \tag{8}\\
\chi_{c}^{*} & =R_{c} \chi^{*}+p_{c}
\end{align*}\right.
$$

with $R_{c}$ and $p_{c}$ the rotation matrix and translation vector from body to camera frames. In accordance with a standard convention in visual servoing, we assume througout the paper that the optical axis of the camera corresponds to the $z$-axis of the camera frame. It follows from (8) that

$$
\begin{align*}
\chi_{c} & =R_{c} \chi+p_{c} \\
& =R_{c}\left[R^{T} \chi^{*}-R^{T} p\right]+p_{c} \\
& =R_{c}\left[R^{T}\left(R_{c}^{T} \chi_{c}^{*}-R_{c}^{T} p_{c}\right)-R^{T} p\right]+p_{c}  \tag{9}\\
& =R_{c} R^{T} R_{c}^{T} \chi_{c}^{*}-\left[R_{c} R^{T} p+\left(R_{c} R^{T} R_{c}^{T}-I_{3}\right) p_{c}\right]
\end{align*}
$$

The homography matrix is thus given by (compare with (5))

$$
\begin{equation*}
H_{c}=R_{c} R^{T} R_{c}^{T}-\frac{1}{d^{*}}\left[R_{c} R^{T} p+\left(R_{c} R^{T} R_{c}^{T}-I_{3}\right) p_{c}\right] n_{c}^{* T} \tag{10}
\end{equation*}
$$

with $n_{c}^{*}=R_{c} n^{*}$ the expression of the normal to the target in the reference camera frame. Approximation of $H_{c}$ around the identity matrix yields

$$
\begin{equation*}
H_{c}=I-S\left(R_{c} \Theta\right)-\frac{1}{d^{*}} R_{c}\left(p+S\left(R_{c}^{T} p_{c}\right) \Theta\right) n_{c}^{* T}+O^{2}(p, \Theta) \tag{11}
\end{equation*}
$$

with $\Theta$ any parametrization of the rotation matrix $R$ such that $R \approx I_{3}+S(\Theta)$ around $R=I_{3}$ (e.g., Euler angles).

## 3 Main results

The results proposed in this section can be viewed as an extension of Proposition 2.1 to System (1). This extension raises several difficulties. First, the control input is no longer the 6 d -vector of velocity variables $(v, \omega)$. It is the 4 d -vector composed of the force input $T$ and torque vector $\Gamma$. Then, one has to account for the system's underactuation, and for the fact that the system's dynamics is not symmetric in all dimensions of the state space : both underactuation and gravity induce differences between vertical versus horizontal dynamics. Such differences are not present when the control design is based on a (holonomic) kinematic model. Finally, we do not assume that measurements of the linear velocity vector $v$ are available.

First, a new error vector which is instrumental in the design of stabilizing feedback laws is defined.

Proposition 3.1 Let $b_{1}, b_{2}, b_{3}$ be the canonical vectors of $\mathbb{R}^{3}$, and $m_{c, k}^{*}=R_{c} b_{k}$ for some $k \in$ $\{1,2,3\}$. Let $n^{*}=\left(n_{1}^{*}, n_{2}^{*}, n_{3}^{*}\right)^{T}$ and $I_{3}$ the identity matrix of size $3 \times 3$. Let (compare with (6))

$$
e^{k}=\binom{e_{p}^{k}}{e_{\Theta}^{k}}, \quad e_{p}^{k}=\left(I-H_{c}\right) m_{c, k}^{*}, \quad e_{\Theta}^{k}=\operatorname{vex}\left(H_{c}^{T}-H_{c}\right)
$$

and

$$
\bar{e}^{k}=A M e^{k}, \quad A=\left(\begin{array}{cc}
R_{c}^{T} & 0_{3}  \tag{12}\\
0_{3} & R_{c}^{T}
\end{array}\right), \quad M=\left(\begin{array}{cc}
2 I_{3} & S\left(m_{c, k}^{*}\right) \\
-S\left(m_{c, k}^{*}\right) & I_{3}
\end{array}\right)
$$

Let $\Theta \in \mathbb{R}^{3}$ denote any parametrization of the rotation matrix $R$ such that $R \approx I_{3}+S(\Theta)$ around $R=I_{3}$ (e.g., Euler angles). Then,

1. In a neighborhood of $(p, R)=\left(0, I_{3}\right)$,

$$
\bar{e}^{k}=L\binom{p+S\left(R_{c}^{T} p_{c}\right) \Theta}{\Theta}+O^{2}(p, \Theta), \quad L=\left(\begin{array}{cc}
L_{p} & 0_{3}  \tag{13}\\
L_{\Theta p} & L_{\Theta}
\end{array}\right)
$$

with

$$
\begin{equation*}
L_{p}=\frac{1}{d^{*}}\left(n_{k}^{*} I_{3}+n^{*} b_{k}^{T}\right), L_{\Theta}=2 I_{3}+S\left(b_{k}\right)^{2}, L_{\Theta p}=\frac{1}{d^{*}} S\left(n^{*}-n_{k}^{*} b_{k}\right) \tag{14}
\end{equation*}
$$

and $O^{2}$ denoting terms of order two at least.
2. If $n_{k}^{*} \neq 0$, then $(p, R) \longmapsto \bar{e}^{k}$ defines a local diffeomorphism around $(p, R)=\left(0, I_{3}\right)$. In particular, $\bar{e}^{k}=0$ if and only if $(p, R)=\left(0, I_{3}\right)$.

The proof is given in Appendix.

Remark 3.2 1) Since the projective vector $m_{c, k}^{*}$ is user-defined, the choice $m_{c, k}^{*}=R_{c} b_{k}$ can always be made. If $m_{c, k}^{*}$ is interpreted as a pointing direction in the camera reference frame, then this direction corresponds to the $b_{k}$ direction in the inertial frame : it is thus an invariant whatever the camera orientation in body frame. 2) Note that $L$ can be viewed as the linear approximation at the origin of the interaction matrix associated with $\bar{e}^{k}$.

Assume that $p_{c}=0$. Then, Eq. (13) shows the rationale behind the definition of $\bar{e}^{k}$ : at first order, components $\bar{e}_{1}^{k}, \bar{e}_{2}^{k}, \bar{e}_{3}^{k}$ contain information on the translation vector $p$ only, while components $\bar{e}_{4}^{k}, \bar{e}_{5}^{k}, \bar{e}_{6}^{k}$ contain decoupled information on the orientation (i.e. $L_{\Theta}$ is diagonal), corrupted by components of the translation vector. This cascade structure is instrumental in the forthcoming control design and analysis. Ensuring that $p_{c}=0$ is not realistic in practice, but fixing the camera close to the center of mass is often possible. The design of exponentially stabilizing feedback laws for $p_{c}=0$ then ensures exponential stability provided that $p_{c}$ is small enough. Therefore, we assume from now on the following.

Assumption : $p_{c}=0$.
The control approach relies on the local diffeomorphism property between $\bar{e}^{k}$ and the pose vector. As shown by Proposition 3.1, however, this property is ensured only if $n_{k}^{*} \neq 0$. For example, when $k=3$ the property is ensured provided that the target is not vertical. Thus, the choice of the value of $k$ has to be made from a priori information on the target. Such information is usually clear from the application context and the camera orientation in body frame. For example, having the camera pointing downward (in body frame) essentially rules out the stabilization w.r.t. a vertical target, while having the camera pointing forward rules out the case of a horizontal target. For completeness, we address below the two main cases of interest.

### 3.1 The case of a ground target

The following result is obtained.
Theorem 3.3 Assume that $n_{3}^{*}>0$ and $\left(R_{c} n^{*}\right)_{3}>0$. Let $\bar{e}_{p} \in \mathbb{R}^{3}$ (resp. $\bar{e}_{\Theta} \in \mathbb{R}^{3}$ ) the first (resp. last) three components of $\bar{e}$, i.e. $\bar{e}=\bar{e}^{3}=\left(\bar{e}_{p}^{T}, \bar{e}_{\Theta}^{T}\right)^{T}$. Let $\bar{e}=\bar{e}^{3}$ and define the control law

$$
\left\{\begin{array}{l}
T=m\left(g+k_{1} \bar{e}_{p_{3}}+k_{2} \nu_{3}\right)  \tag{15}\\
\Gamma=-J K_{3}\left(\omega-\omega^{d}\right)
\end{array}\right.
$$

with

$$
\left\{\begin{align*}
\omega^{d} & =-\frac{K_{4}}{g}\left(g \bar{e}_{\Theta}+b_{3} \times \gamma^{d}\right)  \tag{16}\\
\gamma^{d} & =-K_{5}\left(\bar{e}_{p}+K_{6} \nu\right)
\end{align*}\right.
$$

$\nu$ the variable defined by the dynamic equation

$$
\begin{equation*}
\dot{\nu}=-K_{7} \nu-\bar{e}_{p} \tag{17}
\end{equation*}
$$

Then,

1. Given any $c_{M}^{*}>0$, there exist diagonal gain matrices $K_{i}=\operatorname{Diag}\left(k_{i}^{j}\right) i=3, \ldots, 7 ; j=1,2,3$ and scalar gains $k_{1}, k_{2}$, such that the control law (15) makes the equilibrium $(p, R, v, \omega, \nu)=$ ( $0, I_{3}, 0,0,0$ ) of the closed-loop System (1), (15)-(17) locally exponentially stable provided that

$$
\begin{equation*}
\frac{n_{3}^{*}}{d^{*}} \in\left(0, c_{M}^{*}\right] \tag{18}
\end{equation*}
$$

2. If the diagonal gain matrices $K_{i}$ and scalar gains $k_{1}, k_{2}$ make the closed-loop system locally exponentially stable for $\frac{n_{3}^{*}}{d^{*}}=c_{M}^{*}$, then local exponential stability is guaranteed for any value of $\frac{n_{3}^{*}}{d^{*}} \in\left(0, c_{M}^{*}\right]$.

The proof is given in the Appendix.
Let us comment on this result.

1. The assumption $n_{3}^{*}>0$ means that the target is not vertical and that it is located "below" the UAV (see Figure 1 for illustration), with the camera thus pointing downwards. Since the camera optical axis may differ from the $z$-axis of the body frame, we further impose the visibility assumption of the target at the reference pose : $\left(R_{c} n^{*}\right)_{3}>0$. The case of a ground target located above the UAV, with a camera thus pointing upwards (i.e., $n_{3}^{*}<0$ ), can be addressed similarly, by applying the above control law with $m_{c, k}^{*}=m_{c, 3}^{*}$ replaced by $-m_{c, k}^{*}$ in the definition of $\bar{e}=\bar{e}^{3}$. This is also equivalent to multiplying the first three components of $\bar{e}$ by -1 .
2. The variable $\nu$ copes with the lack of measurements of $\dot{\bar{e}}_{p}$.
3. Since $n_{3}^{*} \leq 1$, (18) is satisfied if $d^{*} \geq 1 / c_{M}^{*}$. Thus, Property 1 ) ensures that stabilizing control gains can be found given any lower bound on the distance between the reference pose and the observed planar target. This is a weak requirement from an application point of view, all the more that this sufficient condition does not involve the (unknown) normal vector $n^{*}$. As for Property 2), it implies that finding stabilizing control gains for any $\frac{n_{3}^{*}}{d^{*}} \in\left(0, c_{M}^{*}\right]$ boils down to finding stabilizing control gains for $\frac{n_{3}^{*}}{d^{*}}=c_{M}^{*}$. This latter task can be easily achieved with classical linear control tools. In particular, by using the Routh-Hurwitz criterion, local exponential stability for $\frac{n_{3}^{*}}{d^{*}}=c_{M}^{*}$ is ensured when the following inequalities are satisfied (see the proof of Theorem 3.3 for details) :

$$
\begin{equation*}
k_{1}, k_{2}, k_{i}^{j}>0, \forall(i, j) \notin\{(5,3),(6,3)\} \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{2}<k_{1} k_{7}^{3} \tag{20}
\end{equation*}
$$

and

$$
\left\{\begin{array}{l}
0<a_{0}^{1}  \tag{21}\\
c_{M}^{*} a_{1}^{1} a_{4}^{1}\left(a_{4}^{1}-a_{0}^{1}\right)<a_{2}^{1} D_{2}^{1} \\
c_{M}^{*} a_{1}^{1}\left(a_{4}^{1}-a_{0}^{1}\right)^{2}<\left(a_{2}^{1}-a_{0}^{1} a_{3}^{1}\right) D_{2}^{1}
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
0<a_{0}^{2}  \tag{22}\\
c_{M}^{*} a_{1}^{2} a_{4}^{2}\left(a_{4}^{2}-a_{0}^{2}\right)<a_{2}^{2} D_{2}^{2} \\
c_{M}^{*} a_{1}^{2}\left(a_{4}^{2}-a_{0}^{2}\right)^{2}<\left(a_{2}^{2}-a_{0}^{2} a_{3}^{2}\right) D_{2}^{2}
\end{array}\right.
$$

with

$$
\begin{gather*}
a_{0}^{1}=k_{7}^{1}-k_{6}^{1}, a_{1}^{1}=k_{3}^{2} k_{4}^{2} k_{5}^{1}, a_{2}^{1}=k_{3}^{2} k_{4}^{2} k_{7}^{1}, \\
a_{3}^{1}=k_{3}^{2}\left(k_{4}^{2}+k_{7}^{1}\right), a_{4}^{1}=k_{3}^{2}+k_{7}^{1}, D_{2}^{1}=a_{4}^{1} a_{3}^{1}-a_{2}^{1} \tag{23}
\end{gather*}
$$

and

$$
\begin{gather*}
a_{0}^{2}=k_{7}^{2}-k_{6}^{2}, a_{1}^{2}=k_{3}^{1} k_{4}^{1} k_{5}^{2}, a_{2}^{2}=k_{3}^{1} k_{4}^{1} k_{7}^{2},  \tag{24}\\
a_{3}^{2}=k_{3}^{1}\left(k_{4}^{1}+k_{7}^{2}\right), a_{4}^{2}=k_{3}^{1}+k_{7}^{2}, D_{2}^{2}=a_{4}^{2} a_{3}^{2}-a_{2}^{2}
\end{gather*}
$$

Let us show the existence of control gains that satisfy the above conditions. First, note that there is no condition on $k_{5}^{3}$ and $k_{6}^{3}$. This is due to the fact that, by (16), these gains do
not affect $\omega^{d}$. Condition (19) is readily satisfied. Control gains $k_{1}, k_{2}, k_{7}^{3}$ are only involved in (20), so that they can be chosen so as to satisfy this inequality. Let us consider (21). First, this set of conditions involves the control gains $k_{3}^{2}, k_{4}^{2}, k_{5}^{1}, k_{6}^{1}$ and $k_{7}^{1}$ only. There are different ways to find values of these gains ensuring (21). We propose next a possibility. Choose any $k_{5}^{1}>0$ and any $k_{6}^{1}, k_{7}^{1}>0$ such that the first inequality in (21) is satisfied. Now set $k_{3}^{2}=k_{4}^{2}=s$. Consider the second inequality in (21). The left-hand side is a polynomial in $s$ with $c_{M}^{*} k_{5}^{1} s^{4}$ as monomial of highest degree. The right-hand side is also a polynomial in $s$ with $k_{7}^{1} s^{5}$ as monomial of highest degree. Choosing $s$ large enough ensures that this inequality is satisfied. The same argument ensures that the last inequality in (21) is satisfied for $s$ large enough. Choosing $k_{3}^{1}, k_{4}^{1}, k_{5}^{2}, k_{6}^{2}$ and $k_{7}^{2}$ in order to satisfy (22) follows the same procedure.
4. Let us finally remark that, given a family of control gains, Conditions (19)-(22) allow to determine the maximum value of $c_{M}^{*}$ for which exponential stability is obtained.

### 3.2 The case of a frontal target

The previous result does not address the case of a vertical target (i.e. $n_{3}^{*}=0$ ). This case is addressed by the following result, which is potentially applicable to any non-horizontal target. The control design uses another error function $\bar{e}$. Without loss of generality we assume that $n_{1}^{*}>0$ ( e.g., the camera is pointing in a direction close to the $b_{1}$ inertial axis at the reference location).

Theorem 3.4 Assume that $n_{1}^{*}>0$ and $\left(R_{c} n^{*}\right)_{3}>0$. Consider the control law (15) -(17) with $\bar{e}=\bar{e}^{1}$ and $\bar{e}_{p} \in \mathbb{R}^{3}$ (resp. $\bar{e}_{\Theta} \in \mathbb{R}^{3}$ ) denoting the first (resp. last) three components of $\bar{e}$, i.e. $\bar{e}=\bar{e}^{1}=\left(\bar{e}_{p}^{T}, \bar{e}_{\Theta}^{T}\right)^{T}$. Then,

1. Given any $c_{M}^{*}>0, \delta \geq 0$, there exist diagonal gain matrices $K_{i}=\operatorname{Diag}\left(k_{i}^{j}\right) i=3, \ldots, 7 ; j=$ $1,2,3$ and scalar gains $k_{1}, k_{2}$, such that the control law (15) makes the equilibrium $(p, R, v, \omega, \nu)=$ $\left(0, I_{3}, 0,0,0\right)$ of the closed-loop System (1), (15)-(17) locally exponentially stable provided that

$$
\begin{equation*}
\frac{1}{d^{*}} \in\left(0, c_{M}^{*}\right] \quad \text { and } \quad \frac{\left|n_{3}^{*}\right|}{\left|n_{1}^{*}\right|} \leq \delta \tag{25}
\end{equation*}
$$

2. If the diagonal gain matrices $K_{i}$ and scalar gains $k_{1}, k_{2}$ make the closed-loop system locally exponentially stable for $\frac{1}{d^{*}}=c_{M}^{*}$ and any $n^{*}$ satisfying (25), then local exponential stability is guaranteed for any value of $\frac{1}{d^{*}} \in\left(0, c_{M}^{*}\right]$.

The proof is given in the Appendix.
Let us comment on this result.

1. As in the case of Theorem 3.3, extension to the case $n_{1}^{*}<0$ can be addressed by applying the control law with $m_{c, k}^{*}=m_{c, 1}^{*}$ replaced by $-m_{c, k}^{*}$ in the definition of $\bar{e}=\bar{e}_{1}$. This is also equivalent to multiplying the first three components of $\bar{e}$ by -1 .
2. The main difference between Theorems 3.3 and 3.4 lies in conditions (18) and (25). Clearly, the latter is more demanding than the former since it cannot be reduced to a condition on $d^{*}$ only. When $n_{3}^{*}=0$ (purely vertical target), however, these conditions are essentially the same.
3. Stability conditions on the control gains are similar to those of Theorem 3.3. More precisely, sufficient conditions of stability for $c^{*}=c_{M}^{*}$ are still given by (19)-(22) with

$$
\begin{gather*}
a_{0}^{1}=k_{7}^{1}-\frac{k_{6}^{1}}{1-g \frac{n_{3}^{*}}{2 k n_{n}^{*}}}, a_{1}^{1}=2 k_{3}^{2} k_{4}^{2} k_{5}^{1}\left(1-g \frac{n_{3}^{*}}{2 k k_{5}^{1} n_{1}^{*}}\right), a_{2}^{1}=k_{3}^{2} k_{4}^{2} k_{7}^{1},  \tag{26}\\
a_{3}^{1}=k_{3}^{2}\left(k_{4}^{2}+k_{7}^{1}\right), a_{4}^{1}=k_{3}^{2}+k_{7}^{1}, D_{2}^{1}=a_{4}^{1} a_{3}^{1}-a_{2}^{1}
\end{gather*}
$$

and

$$
\begin{gather*}
a_{0}^{2}=k_{7}^{2}-k_{6}^{2}, a_{1}^{2}=2 k_{3}^{1} k_{4}^{1} k_{5}^{2}, a_{2}^{2}=4 k_{3}^{1} k_{4}^{1} k_{7}^{2} \\
a_{3}^{2}=4 k_{3}^{1}\left(k_{4}^{1}+k_{7}^{2}\right), a_{4}^{2}=4 k_{3}^{1}+k_{7}^{2}, D_{2}^{2}=a_{4}^{2} a_{3}^{2}-a_{2}^{2} \tag{27}
\end{gather*}
$$

Existence of control gains that satisfy the above conditions for any $n^{*}$ satisfying (25) can be proved as in the case of Theorem 3.3. Note in particular that given the upper-bound in (25) on the ratio $n_{3}^{*} / n_{1}^{*}$, the gain $k_{5}^{1}$ can be chosen so that $0<\tau_{1} \leq 1-g \frac{n_{3}^{*}}{2 k_{5}^{1} n_{1}^{*}} \leq \tau_{2}$ for some constant scalars $\tau_{1}, \tau_{2}$. Then, $k_{6}^{1}, k_{7}^{1}$ can be chosen so that the first inequality in (21) holds. The rest of the gain selection follows as in the case of Theorem 3.3.

## 4 Refinements

Theorem 3.3 provides the theoretical background to stabilize an underactuated VTOL UAV based on videocamera and gyroscope measurements. In practice, however, other issues need to be addressed in the perspective of experiments. One of these issues concerns unmodeled dynamics (like wind). Also, considering stability requirement only is not realistic, since very poor performance would lead to unacceptable behaviour, including possible escape from the stability domain. Thus, gain tuning must be achieved in order to include performance considerations. These two issues are considered next.

### 4.1 Unmodeled dynamics rejection

We consider in this section the presence of unmodeled dynamics acting on the system (like, e.g., wind). Different solutions can be proposed to address this issue, starting with the use of integral correction terms (see, e.g., [14] for more details in a similar context). Due to the simple relation between $\bar{e}_{p}$ given by (12) and $p$, including integral correction terms in the control law of Theorem 3.3 is relatively straightforward. Tuning of the control gains in order to obtain good performance is more difficult, especially for the horizontal dynamics because the distance and normal to the target are unknown. For these dynamics we propose below another approach using accelerometers. We do not provide a complete stability analysis of the proposed solution but simulation results presented further illustrate the approach.

Consider the following model (compare with eq. (1)) :

$$
\left\{\begin{align*}
\dot{p} & =R v  \tag{28}\\
\dot{R} & =R S(\omega) \\
m \dot{v} & =-m S(\omega) v-T b_{3}+m \gamma+R^{T} F_{w} \\
J \dot{\omega} & =-S(\omega) J \omega+\Gamma+\tau_{w} b_{3}
\end{align*}\right.
$$

where $F_{w}$ corresponds to unmodeled translation dynamics, and $\tau_{w}$ to unmodeled yaw dynamics. Both quantities are assumed to be constant. For convenience, we introduce $a_{w}=\frac{F_{w}}{m}$. Recall that
accelerometers measure the following quantity ([22]) :

$$
\begin{equation*}
y_{a c c}=R^{T} \ddot{p}-\gamma=-u b_{3}+R^{T} a_{w} \tag{29}
\end{equation*}
$$

When $a_{w, 1} \neq 0$ or $a_{w, 2} \neq 0, R=I_{3}$ is no longer the equilibrium orientation matrix. We do assume, however, that the new equilibrium orientation matrix is close to the identity matrix so that the approximation $R^{T} a_{w} \approx a_{w}$ is valid. We are aware that this approximation is restrictive but it is justified in the case of "moderate" perturbations. In the case of strong wind, i.e. if the equilibrium orientation is far from the vertical axis, then the attitude angles take large values so that the linear assumption itself is questionable. Also, strong perturbations introduce couplings in the system dynamics which remain to be properly analyzed. For instance, through the term $R^{T} a_{w}$ the horizontal dynamics impacts the vertical one via the term $y_{a c c}$. Closed-loop system analysis in such cases is beyond the objective of this paper.

The following modification of the control law (15)-(16) is proposed :

$$
\left\{\begin{align*}
T & =m\left(g+k_{0} I_{e_{3}}+k_{1} \bar{e}_{3}+k_{2} \nu_{3}\right)  \tag{30}\\
\Gamma & =-J K_{3}\left(\omega-\omega^{d}\right)
\end{align*}\right.
$$

with

$$
\left\{\begin{align*}
\omega^{d} & =-\frac{K_{4}}{g}\left(g \bar{e}_{\Theta}+b_{3} \wedge\left(\gamma^{d}-y_{a c c}\right)+k_{8} I_{e_{6}} b_{3}\right)  \tag{31}\\
\gamma^{d} & =-K_{5}\left(\bar{e}_{p}+K_{6} \nu\right) \\
\dot{I}_{e_{3}} & =\bar{e}_{3} \\
\dot{I}_{e_{6}} & =\bar{e}_{6}
\end{align*}\right.
$$

The idea is to counteract the vertical and yaw unmodelled dynamics by integral terms ( $I_{e_{3}}$ and $I_{e_{6}}$ ), and the horizontal unmodeled dynamics by the accelerometers measurements. Let us briefly analyze the linearized closed-loop dynamics for the control law of Theorem 3.3. For the vertical dynamics one obtains :

$$
\left\{\begin{align*}
\ddot{\bar{e}}_{3} & =2 c^{*}\left(g+a_{w_{3}}-g-k_{0} I_{e_{3}}-k_{1} \bar{e}_{3}-k_{2} \nu_{3}\right)  \tag{32}\\
& =-2 c^{*}\left(k_{0} I_{e_{3}}+k_{1} \bar{e}_{3}+k_{2} \nu_{3}-\left(R^{T} a_{w}\right)_{3}\right) \\
\dot{\nu}_{3} & =-k_{7}^{3} \nu_{3}-\bar{e}_{3}
\end{align*}\right.
$$

with $c^{*}=\frac{n_{3}^{*}}{d^{*}}$. This dynamics is almost the same as in the case of the original control law (see Eq. (43) in the appendix), except for the presence of the projection $\left(R^{T} a_{w}\right)_{3}$ and the integral correction term. Also, the characteristic polynomial -under the simplification $R^{T} a_{w}=a_{w}$ and $a_{w}$ constant - is given by :

$$
\begin{equation*}
\lambda^{4}+k_{7}^{3} \lambda^{3}+2 c^{*}\left[k_{1} \lambda^{2}+\left(k_{0}+k_{1} k_{7}^{3}-k_{2}\right) \lambda+k_{0} k_{7}^{3}\right] \tag{33}
\end{equation*}
$$

Contrary to the original vertical dynamics, this polynomial can go unstable for some values of $c^{*}$, whatever the choice of the gains. More precisely, the Routh-Hurwitz criterion ensures stability if $k_{i}>0, k_{0}+k_{1} k_{7}^{3}-k_{2}>0$ and $2 c^{*} k_{0}^{2}+\left[\left(k_{7}^{3}\right)^{3}+2 c^{*}\left(k_{1} k_{7}^{3}-k_{2}\right)\right] k_{0}-2 c^{*} k_{2}\left(k_{1} k_{7}^{3}-k_{2}\right)<0$. Now, it is always possible to define gains such that this system is stable for some values of $c^{*}$; for instance, chose $k_{1}, k_{7}^{3}, k_{2}>0$ such that $k_{1} k_{7}^{3}-k_{2}>0$ (which is the main stability condition for the original dynamics). Then, one can find $k_{0}>0$ such that the third condition is obtained
for some $c^{*}$ since, for $c^{*}$ fixed, $-2 c^{*} k_{2}\left(k_{1} k_{7}^{3}-k_{2}\right)<0$ so that, as $k_{0}$ decreases towards zero, the condition will be satisfied.

Similarly, upon convergence of $\bar{e}_{3}$ to zero, the horizontal dynamics is given by (compare with (45)) :

$$
\left\{\begin{align*}
\ddot{\bar{e}}_{1} & =c^{*}\left(-g \bar{e}_{5}+y_{a c c_{1}}\right)  \tag{34}\\
\ddot{\bar{e}}_{2} & =c^{*}\left(g \bar{e}_{4}+y_{a c c_{2}}\right) \\
\ddot{\bar{e}}_{4} & =\dot{\omega}_{1}=-k_{3}^{1}\left(\bar{e}_{4}-\omega_{1}^{d}\right) \\
& =-k_{3}^{1}\left(\dot{e}_{4}+\frac{k_{4}^{1}}{g}\left(g \bar{e}_{4}+y_{a c c_{2}}+k_{5}^{2} \bar{e}_{2}+k_{5}^{2} k_{6}^{2} \nu_{2}\right)\right) \\
\ddot{\bar{e}}_{5} & =\dot{\omega}_{2}=-k_{3}^{2}\left(\bar{e}_{5}-\omega_{2}^{d}\right) \\
& =-k_{3}^{2}\left(\dot{e}_{5}+\frac{k_{4}^{2}}{g}\left(g \bar{e}_{5}-y_{a c c_{1}}-k_{5}^{1} \bar{e}_{1}-k_{5}^{1} k_{6}^{1} \nu_{1}\right)\right)
\end{align*}\right.
$$

with $y_{a c c_{i}} \approx a_{w_{i}}$ for $i=1,2$. As a result, the same characteristic polynomials as in the case without perturbation are recovered. Under the same stability conditions on the control gains then, these variables converge asymptotically to zero.

Finally, the yaw dynamics is given by (compare with (52)) :

$$
\begin{equation*}
\ddot{\bar{e}}_{6}=2 \dot{\omega}_{3}=-2 k_{3}^{3}\left(\omega_{3}-\omega_{3}^{d}\right)=-2 k_{3}^{3}\left(\frac{\dot{\bar{e}}_{6}}{2}+k_{4}^{3} \bar{e}_{6}+k_{8} I_{e_{6}}\right)+\left\|\Gamma_{w}\right\| b_{3} \tag{35}
\end{equation*}
$$

Assuming constant perturbation $\left\|\Gamma_{w}\right\|$, the integral term ensures that the yaw error tends to zero if $k_{8}<k_{3}^{3} k_{4}^{3}$.

### 4.2 Gain tuning

While stability is a prerequisite for a closed-loop system, performance cannot be neglected in practice. In particular, it matters to ensure good damping properties. This issue is very important here since we have to cope with a large range of unknown parameters. In this section we propose gain tuning heuristics so as to obtain good performance. These heuristics do not guarantee performance levels, but they have proved effective to obtain good results in simulation. Futhermore, having tuned the gains as proposed, the Barmish theorem [2] can be used to verify performance afterwards. These heuristics are based on the cascade structure of the closed-loop linearized system which allows to address separately the yaw, vertical, and horizontal dynamics (see the proofs of Theorems 3.3 and 3.4 for details). This is similar to the case when full measurement of position, orientation, and velocities is available. We first propose heuristics for the ground target, and then briefly comment on the frontal target case.

### 4.2.1 Yaw dynamics gain tuning :

The caracteristic polynomial associated with the (linearized) yaw dynamics is $P(\lambda)=\lambda^{2}+$ $k_{3}^{3} \lambda+2 k_{3}^{3} k_{4}^{3}$. Thus, any given set of closed-loop poles $\left(\lambda_{1}, \lambda_{2}\right)$ can be assigned by setting

$$
k_{3}^{3}=-\left(\lambda_{1}+\lambda_{2}\right), \quad k_{4}^{3}=-\frac{\lambda_{1} \lambda_{2}}{2\left(\lambda_{1}+\lambda_{2}\right)}
$$

### 4.2.2 Vertical dynamics gain tuning :

The caracteristic polynomial associated with the vertical dynamics is $P(\lambda)=\lambda^{2}\left(\lambda+k_{7}^{3}\right)+$ $C^{*}(\lambda+k)$ with $C^{*}=2 c^{*} k_{1}$ and $k=k_{7}^{3}-\frac{k_{2}}{k_{1}}$. Recall that $c^{*}=\frac{n_{3}^{*}}{d^{*}}$. The following heuristic is proposed :

1. Define the gain $k_{7}^{3}$ and a number $k \neq k_{7}^{3}$ knowing that, as $c^{*}$ grows from 0 to $\infty$, the closed loop gains will move from 0 and $-k_{7}^{3}$ to $-k$ and $\frac{k-k_{7}^{3}}{2}$.
2. The slowest poles' real parts will start from 0 and head to $\frac{k-k_{7}^{3}}{2}$ : define the scaling factor $k_{1}$ so as to define $c_{\min }^{*}$ for which a given real part is reached. Note that $k_{2}$ is then given by : $k_{2}=k_{1}\left(k_{7}^{3}-k\right)$.
3. Use [2] to assess the performance of the obtained closed-loop system as $c^{*}$ varies in its allowed range.

Justification : The root locus theory shows that the poles will start from $\left(-k_{7}^{3}, 0,0\right)$ as $c^{*}=0$ and head to $-k$ and the two asymptotic directions $\frac{k-k_{7}^{3}}{2} \pm j \infty$ as $c^{*} \rightarrow \infty$. One can also verify that, whatever the gains such that $k \neq k_{7}^{3}$, there is no root on the imaginary axis.
Numerical example : With $k_{1}=5, k_{2}=10, k_{7}^{3}=2.4$, the root locus shows that:

1. $\forall c^{*}>0, \Re\left(\lambda_{i}\right)<0$
2. $\forall c^{*} \in[0.175 ;+\infty],-1 \leq \Re\left(\lambda_{i}\right) \leq-0.4$
3. $\forall c^{*} \in[0.175 ; 2.34], \xi \geq 0.2$ ( $\xi$ is the damping ratio)

### 4.2.3 Horizontal dynamics gain tuning :

The horizontal dynamics is composed of two fifth-order linear systems (47)-(48). Since the structure of these systems is the same we only address gain tuning for the first one. The associated characteristic polynomial is :

$$
\lambda^{2}\left(\lambda+k_{7}^{1}\right)\left(\lambda^{2}+k_{3}^{2} \lambda+k_{3}^{2} k_{4}^{2}\right)+C^{*}[\lambda+K]
$$

where $C=c^{*} k_{3}^{2} k_{4}^{2} k_{5}^{1}, K=k_{7}^{1}-k_{6}^{1}$. The following heuristics is proposed :

1. Select $k_{3}^{2}$ and $k_{4}^{2}$ such that the roots of $\lambda^{2}+k_{3}^{2} \lambda+k_{3}^{2} k_{4}^{2}$ are as fast as possible;
2. Define a much slower dynamics for the "inside system" defined by : $k_{3}^{2} k_{4}^{2} \lambda^{2}\left(\lambda+k_{7}^{1}\right)+$ $C^{*}[\lambda+K]$, and select suitable $k_{5}^{1}, k_{6}^{1}, k_{7}^{1}$ so that for $c^{*} \in\left(0 ; c_{M}^{*}\right]$, all poles are slower than the above defined maximum inside dynamics ;
3. Use [2] to assess the performance of the obtained closed-loop system as $c^{*}$ varies in its allowed range.

Justification : From the root locus theory there are poles at 0 (double), $-k_{7}^{1}$ and at the roots of $\lambda^{2}+k_{3}^{2} \lambda+k_{3}^{2} k_{4}^{2}$. The only zero is at $-\left(k_{7}^{1}-k_{6}^{1}\right)$ (zero of the inside dynamics). The two poles placed by $k_{3}^{2}$ and $k_{4}^{2}$ will go to infinity as $c^{*}$ grows. The poles at 0 and $-k_{7}^{1}$ will behave similarly to the vertical dynamics for small $c^{*}$, since they are close to zero and separated from the first two ; for $c^{*}$ large, two of these poles will escape to infinity with positive real part.

Numerical Example : With gains defined as $k_{3}^{2}=10, k_{4}^{2}=12, k_{5}^{1}=5, k_{6}^{1}=2, k_{7}^{1}=2.4$ (inside dynamics being slower than -1 , which is must slower than the roots of $\lambda^{2}+k_{3}^{2} \lambda+k_{3}^{2} k_{4}^{2}$ ), the caracteristic polynomial is given by : $\lambda^{5}+12.4 \lambda^{4}+144 \lambda^{3}+288 \lambda^{2}+600 c^{*} \lambda+240 c^{*}$, which is stable for $c^{*} \in\left(0 ; c_{M}^{*}\right]$ with $c_{M}^{*} \approx 4$, and such that the roots real parts $\Re(\lambda) \leq-0.2$ for $c^{*} \in\left[c_{1}^{*} ; c_{2}^{*}\right]$ with $c_{1}^{*} \approx 0.2$ and $c_{2}^{*} \approx 3.25$.

For the simulations results reported next, the gain values were obtained through this procedure.

Remark on the frontal target case : For the control law of Theorem 3.4, almost the same heuristics can be used since the equations are similar. Stability conditions, however, involve the unknown ratio $n_{3}^{*} / n_{1}^{*}$ (see (26)). As already mentionned in Section 3.2, choosing $k_{5}^{1}$ large makes the root-locus less sensitive to this ratio. Then, $k_{3}^{2}$ and $k_{4}^{2}$ must also be chosen large enough so as to satisfy the stability condition (21). For instance, satisfactory results have been obtained by changing as follows the three gains $k_{5}^{1}, k_{3}^{2}, k_{4}^{2}$ from their values designed for the ground target case : $k_{5}^{1}$ multiplied by $5, k_{3}^{2}, k_{4}^{2}$ multiplied by 4 . For this choice, expressing $\frac{n_{3}}{n_{1}}$ as the tangent of an angle $\alpha$ representing the verticality of the target, one obtains the following results :

- with $\alpha=0^{\circ}, c_{M}^{*}=1.8$ and the root locus is similar to the ground target case except for the $k_{3}^{2}, k_{4}^{2}$ dynamics, which is much faster ;
- with $\alpha=-30^{\circ}, c_{M}^{*}=1.3$;
- with $\alpha=+30^{\circ}, c_{M}^{*}=2.1$ but the slow dynamics is quite different, so that there is, for any value of $c^{*}$ a slow dynamics around the new zero defined by $-a_{0}^{1}$.


## 5 Simulation results

We present simulation results for the basic control law (Theorems 3.3 and 3.4) and for its modified version proposed in Section 4.1. These results have been obtained for a dynamical model of an helicopter defined by (1) with $m=10$ and $J=I_{3}$ the identity matrix. Simulations are presented for different values of the distance to the target at the reference pose, with initial position errors equal to : $p_{0}=(1.5,-1,0.5)^{T}$, and the other initial conditions (orientation error, linear and angular velocity) null. $\phi, \theta, \psi$ are respectively the roll, pitch and yaw angles.

### 5.1 Control law (15)-(17) of Section 3

Case of a ground target : Simulation results with the control law of Theorem 3.3 are reported on Fig. 3-4. The normal vector $n^{*}$ which defines the target orientation was chosen as $n^{*}=$ $(-0.28,0.28,0.92)^{T}$. Simulations results with $d^{*}=2$ and $d^{*}=10$ are reported. This corresponds to a very large range of nominal distances to the target. The rotation matrix $R_{c}$ which defines the camera orientation in body frame was randomly generated, under the constraint that the associated rotation angle is equal to $15^{\circ}$. To test the robustness of the approach, a position offset $p_{c}$ of the camera frame w.r.t. the body frame was also introduced. The values of $R_{c}$ and $p_{c}$ are the following :

$$
R_{c}=\left(\begin{array}{ccc}
0.97 & 0.20 & 0.16 \\
-0.19 & 0.98 & -0.06 \\
-0.17 & 0.03 & 0.99
\end{array}\right) \quad p_{c}=(0.30,-0.20,0.10)^{T}
$$

The control gains in (15)-(16) have been chosen as follows, according to the gain tuning heuristics described above :

$$
\begin{align*}
& k_{1}=5, k_{2}=10, K_{3}=\operatorname{Diag}\left(10,10, \frac{\sqrt{2}}{2}\right), K_{4}=\operatorname{Diag}(12,12,4)  \tag{36}\\
& K_{5}=\operatorname{Diag}(5,5,0), K_{6}=\operatorname{Diag}(2,2,0), K_{7}=\operatorname{Diag}(2.05,2.05,2.5)
\end{align*}
$$

From (21)-(22), this yields the stability upper-bound $c_{M}^{*}<3.99$. Since $n_{3}^{*}=0.92$, this yields $d_{M}^{*}>0.23$, which is consistent with the simulation values $d^{*}=2$ and $d^{*}=10$. Fig. 3-4 illustrate
the capacity of the controller to perform well in all this range, without any depth or target's orientation information.
Case of a frontal target : Simulation results with the control law of Theorem 3.4 are reported on Fig. $5-6$. The normal vector $n^{*}$ is now given by $n^{*}=(0.93,0.16,-0.34)^{T}$. Results for $d^{*}=2$ and $d^{*}=10$ are reported, with the following choice of $R_{c}$ and $p_{c}$ :

$$
R_{c}=\left(\begin{array}{ccc}
-0.21 & 0.97 & -0.15 \\
-0.07 & 0.13 & 0.99 \\
0.98 & 0.22 & 0.04
\end{array}\right) \quad p_{c}=(0.50,0.30,-0.05)^{T}
$$

The control gains are still given by (36). Performances are similar to those obtained for the previous set of simulations.

### 5.2 Control law (30)-(31) of Section 4.1 for unmodeled dynamics rejection

In this section we compare the performances of the original control law (15)-(17) and the modified control law (30)-(31) in the presence of wind, with the same simulation parameters as for Fig. 3-4. For both controllers the control gains (36) have been used. For the control law (30)-(31), the integral gain on the vertical dynamics is defined as $k_{0}=1$. From the expression of the vertical dynamics' characteristic polynomial, this value ensures stability as long as $c^{*}>0.36$, or $d^{*}<2.53$ in this case. The wind effect has been modeled as a drag force (no lift). More precisely, a constant or slowly-time varying wind velocity $\dot{p}_{w}$ has been used to compute the relative wind velocity : $\delta \dot{p}=\dot{p}_{w}-\dot{p}$, with $\dot{p}_{w}$ the wind velocity. The drag force is then given by $F_{w}=-\frac{1}{2} C_{x} \rho \Sigma\|\delta \dot{p}\| \delta \dot{p}$ where the total drag coefficient $k_{w}=\frac{1}{2} C_{x} \rho \Sigma$ has been chosen equal to 0.1 . This is a realistic value given the mass and inertia of the system. Fig. 7 and 8 have been obtained with the control laws (15)-(17) and (30)-(31) respectively, with the constant wind velocity $v_{w}=[-2.50,3.00,1.50] \mathrm{m} / \mathrm{s}$ and the distance to the target at the reference pose $d^{*}=2$ (i.e. $c^{*}=0.46$ ). The efficiency of the modified controller for wind effect rejection is clearly illustrated.

The case of a slowly varying wind has also been considered, according to the following random walk model : $\ddot{p}_{w}=k_{w} \xi$ with $\dot{p}_{w}(0)=[-2.50,3.00,1.50] m / s, k_{w}=0.3$ and $\xi$ a centered white noise with spectrum one. Again, Fig. 9 and 10 clearly show that the modified controller outperforms the original one.

## 6 Conclusion

Feedback laws have been proposed for the vision-based stabilization of VTOL UAVs w.r.t. a planar target. The main contribution of this work is to show that such a stabilization can be achieved with a minimal sensor suite (a mono-camera and gyrometers), and with very poor knowledge about the environment. In particular, precise knowledge of the target's orientation or distance from the UAV to the target at the reference pose are not necessary. Knowledge of the UAV's attitude is not necessary either. Explicit stability conditions derived in this paper can be used to guarantee stability of the proposed controllers for a range of operational conditions. The approach, motivated by [3], relies on the definition of a new homography-based error vector. By using this vector, the classical (local) decoupling of vertical, roll, pitch, and yaw dynamics can be extended to the vision-based control framework. In addition to the control design and stability analysis, several practical issues have also been addressed like the influence of the camera's
orientation in the control law definition, unmodeled dynamics rejection, and gain tuning. The approach has been validated in simulation. Extensions of this work include experimental validations on mini-drones, and nonlinear control design in order to possibly extend the stability domain and allow more aggressive manoeuvers.

## Appendix

Proof of Proposition 3.1 : Recall from (11) that

$$
\begin{equation*}
H_{c}=I-S\left(R_{c} \Theta\right)-\frac{1}{d^{*}} R_{c} \bar{p} n_{c}^{* T}+O^{2}(p, \Theta) \tag{37}
\end{equation*}
$$

with $\bar{p}=p+S\left(R_{c}^{T} p_{c}\right) \Theta$. Using the fact that $n_{c}^{*}=R_{c} n^{*}$ and $m_{c, k}^{*}=R_{c} b_{k}$, we deduce from (6) and (37) that

$$
\begin{align*}
e_{p} & =\frac{1}{d^{*}}\left\langle n_{c}^{*}, m_{c, k}^{*}\right\rangle R_{c} \bar{p}+S\left(R_{c} \Theta\right) m_{c, k}^{*}+O^{2}(p, \Theta) \\
& =R_{c}\left(\frac{n_{k}^{*}}{d^{*}} \bar{p}-S\left(b_{k}\right) \Theta\right)+O^{2}(p, \Theta)  \tag{38}\\
e_{\Theta} & =2 R_{c} \Theta+\frac{1}{d^{*}} S\left(n_{c}^{*}\right) R_{c} \bar{p}+O^{2}(p, \Theta) \\
& =R_{c}\left(2 \Theta+\frac{1}{d^{*}} S\left(n^{*}\right) \bar{p}\right)+O^{2}(p, \Theta)
\end{align*}
$$

where we have used the fact that $x y^{T}-y x^{T}=S(S(y) x), \forall x, y \in \mathbb{R}^{3}$. We deduce from (38) that

$$
\begin{align*}
2 R_{c}^{T} e_{p}+R_{c}^{T} S\left(m_{c, k}^{*}\right) e_{\Theta} & =2 R_{c}^{T} e_{p}+S\left(b_{k}\right) R_{c}^{T} e_{\Theta} \\
& =\frac{1}{d^{*}}\left(2 n_{k}^{*} I_{3}+S\left(b_{k}\right) S\left(n^{*}\right)\right) \bar{p}+O^{2}(p, \Theta)  \tag{39}\\
& =\frac{1}{d^{*}}\left(n_{k}^{*} I_{3}+n^{*} b_{k}^{T}\right) \bar{p}+O^{2}(p, \Theta)
\end{align*}
$$

where the last equality comes from the fact that $x \times(y \times z)=y\left(x^{T} z\right)-z\left(x^{T} y\right)$. We also deduce from (38) that

$$
\begin{align*}
R_{c}^{T} e_{\Theta}-R_{c}^{T} S\left(m_{c, k}^{*}\right) e_{p} & =R_{c}^{T} e_{\Theta}-S\left(b_{k}\right) R_{c}^{T} e_{p}+O^{2}(p, \Theta) \\
& =\frac{1}{d^{*}} S\left(n^{*}-n_{k}^{*} b_{k}\right) \bar{p}+\left(2 I_{3}+S\left(b_{k}\right)^{2}\right) \Theta+O^{2}(p, \Theta) \tag{40}
\end{align*}
$$

Relation (13) follows from (39) and (40).
Finally, the second property in Proposition 3.1 follows directly form the block-triangular structure of $L$, and the (easily verified) fact that $L_{p}$ is invertible when $n_{k}^{*} \neq 0$ and $L_{\Theta}$ is always invertible.

Proof of Theorem 3.3: By definition, $R \approx I_{3}+S(\Theta)$ around $R=I_{3}$. Therefore, $\gamma=g R^{T} b_{3} \approx$ $g b_{3}+g S\left(b_{3}\right) \Theta$ around $R=I_{3}$ and it follows from (1) and (15)- (17) that the linearized controlled system around $(p, R, v, \omega, \nu)=\left(0, I_{3}, 0,0,0\right)$ is given by

$$
\left\{\begin{align*}
\ddot{p} & =g S\left(b_{3}\right) \Theta+\left(g-\frac{T}{m}\right) b_{3}  \tag{41}\\
\ddot{\Theta} & =J^{-1} \Gamma \\
\dot{\nu} & =-K_{7} \nu-\bar{e}_{p}
\end{align*}\right.
$$

Note that in the above (and forthcoming) equations, only the first-order linear approximations of $T, \Gamma$, and $\bar{e}_{p}$ should be considered (i.e., the expressions obtained by omitting in (13) the term $O^{2}(p, \Theta)$. To lighten the notation these linear approximations are still denoted as $T, \Gamma$, and $\bar{e}_{p}$. Finally, we denote $c^{*}=\frac{n_{3}^{*}}{d^{*}}$.

We now study the stability of the equilibrium $(p, \Theta, \dot{p}, \dot{\Theta}, \nu)=0$. It follows from (13) and (14) that the linearized controlled system in the coordinates $(\bar{e}, \dot{\bar{e}}, \nu)$ is given by

$$
\left\{\begin{array}{l}
\ddot{\bar{e}}=L\binom{\ddot{p}}{\ddot{\Theta}}=L\binom{g S\left(b_{3}\right) \Theta+\left(g-\frac{T}{m}\right) b_{3}}{J^{-1} \Gamma}  \tag{42}\\
\dot{\nu}=-K_{7} \nu-\bar{e}_{p}
\end{array}\right.
$$

wth the components of $L$ given by

$$
L_{p}=c^{*}\left(\begin{array}{ccc}
1 & 0 & \frac{n_{1}^{*}}{n_{3}^{*}} \\
0 & 1 & \frac{n_{2}^{2}}{n_{3}^{*}} \\
0 & 0 & 2
\end{array}\right), L_{\Theta}=\operatorname{Diag}(1,1,2), L_{\Theta p}=\frac{1}{d^{*}}\left(\begin{array}{ccc}
0 & 0 & n_{2}^{*} \\
0 & 0 & -n_{1}^{*} \\
-n_{2}^{*} & n_{1}^{*} & 0
\end{array}\right)
$$

There remains to determine conditions under which $(\bar{e}, \dot{\bar{e}}, \nu)=0$ is an asymptotically stable equilibrium of this system. We proceed in three steps, in which the convergence to zero of the vertical, horizontal, and yaw variables are successively studied.

Step 1 : It follows from the above relations and (15)-(17) that

$$
\left\{\begin{array}{l}
\ddot{\bar{e}}_{3}=2 c^{*}(g-T / m)=-2 c^{*}\left(k_{1} \bar{e}_{3}+k_{2} \nu_{3}\right)  \tag{43}\\
\dot{\nu}_{3}=-k_{7}^{3} \nu_{3}-\bar{e}_{3}
\end{array}\right.
$$

Thus the dynamics of $\bar{e}_{3}, \dot{\bar{e}}_{3}, \nu_{3}$ is independent of the other variables and the origin of the above system is asymptotically stable provided that the characteristic polynomial of the system's state matrix is Hurwitz-Stable. This polynomial is given by

$$
P(\lambda)=\lambda^{3}+k_{7}^{3} \lambda^{2}+2 c^{*} k_{1} \lambda+2 c^{*}\left(k_{1} k_{7}^{3}-k_{2}\right)
$$

Considering that $c^{*}>0$, application of the Routh-Hurwitz criterion yields that $P$ is HurwitzStable if and only if

$$
\begin{equation*}
k_{1}, k_{2}, k_{7}^{3}>0, k_{1} k_{7}^{3}>k_{2} \tag{44}
\end{equation*}
$$

Note that this condition is independent of $c^{*}(>0)$.
Step 2 : Under the condition that $\bar{e}_{3}, \dot{\bar{e}}_{3}$, and $\nu_{3}$ converge asymptotically to zero, we can concentrate on the zero-dynamics $\left(\bar{e}_{3}, \dot{\bar{e}}_{3}, \nu_{3}\right)=0$. From (13), $\bar{e}_{3}=0$ implies that $p_{3}=0$ for the linearized equations so that,

$$
\binom{\bar{e}_{1}}{\bar{e}_{2}}=c^{*}\binom{p_{1}}{p_{2}}, \quad\binom{\bar{e}_{4}}{\bar{e}_{5}}=\binom{\Theta_{1}}{\Theta_{2}}
$$

This implies, using (41) and the expression (15) of $\Gamma$, that

$$
\left\{\begin{align*}
\ddot{\bar{e}}_{1} & =-c^{*} g \bar{e}_{5}  \tag{45}\\
\ddot{\bar{e}}_{2} & =c^{*} g \bar{e}_{4} \\
\ddot{\bar{e}}_{4} & =\dot{\omega}_{1}=-k_{3}^{1}\left(\dot{\bar{e}}_{4}-\omega_{1}^{d}\right) \\
& =-k_{3}^{1}\left(\dot{\bar{e}}_{4}+\frac{k_{4}^{1}}{g}\left(g \bar{e}_{4}+k_{5}^{2} \bar{e}_{2}+k_{5}^{2} k_{6}^{2} \nu_{2}\right)\right) \\
\ddot{\ddot{e}}_{5} & =\dot{\omega}_{2}=-k_{3}^{2}\left(\overline{\bar{e}}_{5}-\omega_{2}^{d}\right) \\
& =-k_{3}^{2}\left(\dot{\bar{e}}_{5}+\frac{k_{4}^{2}}{g}\left(g \bar{e}_{5}-k_{5}^{1} \bar{e}_{1}-k_{5}^{1} k_{6}^{1} \nu_{1}\right)\right)
\end{align*}\right.
$$

From (17),

$$
\left\{\begin{array}{l}
\dot{\nu}_{1}=-k_{7}^{1} \nu_{1}-\bar{e}_{1}  \tag{46}\\
\dot{\nu}_{2}=-k_{7}^{2} \nu_{2}-\bar{e}_{2}
\end{array}\right.
$$

System (45)-(46) can be decomposed into two independent 5-th order linear systems :

$$
\left\{\begin{array}{l}
\ddot{\bar{e}}_{1}=-c^{*} g \bar{e}_{5}  \tag{47}\\
\ddot{\bar{e}}_{5}=-k_{3}^{2}\left(\dot{\bar{e}}_{5}+\frac{k_{4}^{2}}{g}\left(g \bar{e}_{5}-k_{5}^{1} \bar{e}_{1}-k_{5}^{1} k_{6}^{1} \nu_{1}\right)\right) \\
\dot{\nu}_{1}=-k_{7}^{1} \nu_{1}-\bar{e}_{1}
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\ddot{\bar{e}}_{2}=c^{*} g \bar{e}_{4}  \tag{48}\\
\ddot{\bar{e}}_{4}=-k_{3}^{1}\left(\dot{\bar{e}}_{4}+\frac{k_{4}^{1}}{g}\left(g \bar{e}_{4}+k_{5}^{2} \bar{e}_{2}+k_{5}^{2} k_{6}^{2} \nu_{2}\right)\right) \\
\dot{\nu}_{2}=-k_{7}^{2} \nu_{2}-\bar{e}_{2}
\end{array}\right.
$$

Let us analyze the stability of these systems. The characteristic polynomial of System (47) is

$$
\begin{align*}
P_{c^{*}}^{1}(\lambda) & =\lambda^{2}\left(\lambda+k_{7}^{1}\right)\left(\lambda^{2}+k_{3}^{2} \lambda+k_{3}^{2} k_{4}^{2}\right)+c^{*} a_{1}^{1}\left[\lambda+a_{0}^{1}\right] \\
& =\lambda^{5}+a_{4}^{1} \lambda^{4}+a_{3}^{1} \lambda^{3}+a_{2}^{1} \lambda^{2}+c^{*} a_{1}^{1} \lambda+c^{*} a_{1}^{1} a_{0}^{1} \tag{49}
\end{align*}
$$

with the $a_{i}^{1}$ 's defined by (23). Similarly, the characteristic polynomial of System (48) is

$$
\begin{equation*}
P_{c^{*}}^{2}(\lambda)=\lambda^{5}+a_{4}^{2} \lambda^{4}+a_{3}^{2} \lambda^{3}+a_{2}^{2} \lambda^{2}+c^{*} a_{1}^{2} \lambda+c^{*} a_{1}^{2} a_{0}^{2} \tag{50}
\end{equation*}
$$

with the $a_{i}^{2}$ 's defined by (24). Both polynomials are of the form

$$
P_{c^{*}}^{j}(\lambda)=\lambda^{5}+a_{4}^{j} \lambda^{4}+a_{3}^{j} \lambda^{3}+a_{2}^{j} \lambda^{2}+c^{*} a_{1}^{j} \lambda+c^{*} a_{1}^{j} a_{0}^{j}
$$

Let us determine necessary and sufficient conditions for the Hurwitz-stability of such polynomials. It is well known (and easy to verify) that a necessary stability condition is that all coefficients are strictly positive. Since $c^{*}>0$, this implies that all $a_{i}^{j}$ must be strictly positive. By application of the Routh-Hurwitz criterion, we obtain the following necessary and sufficient condition for stability of $P_{c^{*}}^{j}$ :

$$
\left\{\begin{array}{l}
a_{i}^{j}>0, \quad \forall i \\
D_{2}^{j}=a_{4}^{j} a_{3}^{j}-a_{2}^{j}>0 \\
D_{3}^{j}=a_{2}^{j} D_{2}^{j}-c^{*} a_{4}^{j} a_{1}^{j}\left(a_{4}^{j}-a_{0}^{j}\right)>0 \\
D_{4}^{j}=c^{*} a_{1}^{j}\left[D_{3}^{j}-a_{0}^{j}\left(a_{3}^{j} D_{2}^{j}-c^{*} a_{1}^{j}\left(a_{4}^{j}-a_{0}^{j}\right)\right)\right]>0
\end{array}\right.
$$

This set of inequalities is equivalent to :

$$
\left\{\begin{array}{l}
a_{i}^{j}>0, \quad \forall i  \tag{51}\\
D_{2}^{j}=a_{4}^{j} a_{3}^{j}-a_{2}^{j}>0 \\
c^{*} a_{4}^{j} a_{1}^{j}\left(a_{4}^{j}-a_{0}^{j}\right)<a_{2}^{j} D_{2}^{j} \\
c^{*} a_{1}^{j}\left(a_{4}^{j}-a_{0}^{j}\right)^{2}<\left(a_{2}^{j}-a_{0}^{j} a_{3}^{j}\right) D_{2}^{j}
\end{array}\right.
$$

Since $a_{1}^{j}$ must be strictly positive, the last inequality in (51) is satisfied for any $c^{*} \in\left(0, c_{M}^{*}\right]$ if and only if it is satisfied for the maximum value $c_{M}^{*}$. Let us consider the last but one inequality. If $a_{4}^{j}-a_{0}^{j}>0$, by a similar argument as above, this inequality is satisfied for any $c^{*} \in\left(0, c_{M}^{*}\right]$
if and only if it is satisfied for $c_{M}^{*}$. If $a_{4}^{j}-a_{0}^{j} \leq 0$, it is satisfied for any $c^{*}>0$ due to the fact that $a_{4}^{j}, a_{1}^{j}, a_{2}^{j}, D_{2}^{j}>0$. To summarize, we have shown that the polynomials $P_{c^{*}}^{j}, j=1,2$ are Hurwitz-stable for any $c^{*} \in\left(0, c_{M}^{*}\right]$ if and only if they are Hurwitz-stable for $c_{M}^{*}$, with the stability conditions given by (51) for $c^{*}=c_{M}^{*}$.

Step 3: Assuming the convergence to zero of $\bar{e}_{1}, \ldots, \bar{e}_{5}, \dot{\bar{e}}_{1}, \ldots, \dot{\bar{e}}_{5}$, and $\nu$, let us consider the variables $\bar{e}_{6}, \dot{\bar{e}}_{6}$. It follows from (13) and (42) that on the zero-dynamics $\bar{e}_{1}=\ldots=\bar{e}_{5}=0$,

$$
\begin{equation*}
\ddot{\bar{e}}_{6}=2 \dot{\omega}_{3}=-2 k_{3}^{3}\left(\omega_{3}-\omega_{3}^{d}\right)=-2 k_{3}^{3}\left(\frac{\dot{\bar{e}}_{6}}{2}+k_{4}^{3} \bar{e}_{6}\right) \tag{52}
\end{equation*}
$$

The dynamics of this second-order linear system is asymptotically stable if and only if

$$
\begin{equation*}
k_{3}^{3}, k_{4}^{3}>0 \tag{53}
\end{equation*}
$$

To summarize, we have shown that the subsystems (43), (47), (48), and (52) are asymptotically stable for any $c^{*} \in\left(0, c_{M}^{*}\right]$ if and only if they are asymptotically stable for $c^{*}=c_{M}^{*}$. In view of the cascade structure of the linearized system (42), it is easy to verify that asymptotic stability of these subsystems is necessary and sufficient for the asymptotic stability of (42), and thus for the local exponential stability of the original nonlinear system. This concludes the proof of Property 2) of the Theorem.

As for Property 1), one observes that Conditions (19), (21), (22) imply the stability conditions (44), (51) for $j=1,2$ and $c^{*}=c_{M}^{*}$, and (53). Existence of control gains satisfying (19), (21), (22) has been shown in Section 3.

Proof of Theorem 3.4: The expression (41) of the linearized system is still valid in this case but the components of $L$ are now given by (see (14) with $k=1$ )

$$
L_{p}=c^{*}\left(\begin{array}{ccc}
2 & 0 & 0 \\
\frac{n_{2}^{*}}{n_{*}^{*}} & 1 & 0 \\
\frac{n_{3}^{*}}{n_{1}^{*}} & 0 & 1
\end{array}\right), L_{\Theta}=\operatorname{Diag}(2,1,1), L_{\Theta p}=\frac{1}{d^{*}}\left(\begin{array}{ccc}
0 & -n_{3}^{*} & n_{2}^{*} \\
n_{3}^{*} & 0 & 0 \\
-n_{2}^{*} & 0 & 0
\end{array}\right)
$$

where $c^{*}=\frac{n_{1}^{*}}{d^{*}}$.
Step 1 : It follows from the above expression and (15)-(17), (41) that

$$
\left\{\begin{array}{l}
\bar{e}_{1}=2 c^{*} p_{1} \\
\ddot{\bar{e}}_{1}=-2 c^{*} g \Theta_{2} \\
\ddot{\Theta}_{2}=-k_{3}^{2}\left(\dot{\Theta}_{2}-\omega_{2}^{d}\right) \\
\omega_{2}^{d}=-\frac{k_{9}^{2}}{9}\left(g \bar{e}_{5}-k_{5}^{1} \bar{e}_{1}-k_{5}^{1} k_{6}^{1} \nu_{1}\right) \\
\dot{\nu}_{1}=-k_{7}^{1} \nu_{1}-\bar{e}_{1}
\end{array}\right.
$$

We deduce from these expressions, after a few calculations, that the characteristic polynomial associated with the dynamics of $\bar{e}_{1}$ is given by (49) with the $a_{i}^{1}$ 's defined by (26). Following the proof of Theorem 3.3, and using the fact that $c^{*}=n_{1}^{*} / d^{*}$ and $d^{*}$ is not involved in the definition of the $a_{i}^{j}$ 's, this polynomial is Hurwitz-stable for $1 / d^{*} \in\left(0, c_{M}^{*}\right]$ if and only if it is Hurwitz-stable for $1 / d^{*}=c_{M}^{*}$, with the stability condition given by (51) for $j=2$.
Step 2 : Under the condition that $\bar{e}_{1}, \bar{e}_{5}\left(=\Theta_{2}+\frac{n_{2}^{*}}{2 n_{1}^{*}} \bar{e}_{1}\right)$, and $\nu_{1}$ converge to zero, one has on the zero-dynamics $\bar{e}_{3}=c^{*} p_{3}$. It follows from this relation and (15)-(17) that (compare with (43))

$$
\begin{cases}\ddot{\bar{e}}_{3} & =c^{*}(g-T / m)=-c^{*}\left(k_{1} \bar{e}_{3}+k_{2} \nu_{3}\right)  \tag{54}\\ \dot{\nu}_{3}=-k_{7}^{3} \nu_{3}-\bar{e}_{3}\end{cases}
$$

The characteristic polynomial associated with this linear system is

$$
P(\lambda)=\lambda^{3}+k_{7}^{3} \lambda^{2}+c^{*} k_{1} \lambda+2 c^{*}\left(k_{1} k_{7}^{3}-k_{2}\right)
$$

Considering that $c^{*}>0$, application of the Routh-Hurwitz criterion yields that $P$ is HurwitzStable if and only if Condition (44) is satisfied.

Step 3 : Under the condition that $\bar{e}_{1}, \bar{e}_{3}, \bar{e}_{5}, \nu_{1}$, and $\nu_{3}$ converge to zero, one has on the zero dynamics $\bar{e}_{2}=c^{*} p_{2}$ and $\bar{e}_{4}=2 \Theta_{1}$. The following dynamics for $\bar{e}_{2}, \bar{e}_{4}$ is obtained (compare with (48)) :

$$
\left\{\begin{array}{l}
\ddot{\ddot{e}}_{2}=c^{*} g \bar{e}_{4} / 2  \tag{55}\\
\ddot{\ddot{e}}_{4}=-2 k_{3}^{1}\left(\dot{\bar{e}}_{4}+\frac{k_{4}^{1}}{g}\left(g \bar{e}_{4}+k_{5}^{2} \bar{e}_{2}+k_{5}^{2} k_{6}^{2} \nu_{2}\right)\right) \\
\dot{\nu}_{2}=-k_{7}^{2} \nu_{2}-\bar{e}_{2}
\end{array}\right.
$$

The characteristic polynomial associated with this system is given by (50) with the $a_{i}^{2}$ 's defined by (27). Using the same arguments as above, we deduce that this polynomial is Hurwitz-stable for $1 / d^{*} \in\left(0, c_{M}^{*}\right]$ if and only if it is asymptotically stable for $1 / d^{*}=c_{M}^{*}$, with the stability condition given by (51) for $j=1$.

Step 4 : Finally, on the zero-dynamics $\bar{e}_{1}=0$, we also have $\bar{e}_{6}=\Theta_{3}$ so that (compare with (52)),

$$
\ddot{\bar{e}}_{6}=\dot{\omega}_{3}=-k_{3}^{3}\left(\omega_{3}-\omega_{3}^{d}\right)=-k_{3}^{3}\left(\dot{\bar{e}}_{6}+k_{4}^{3} \bar{e}_{6}\right)
$$

The dynamics of this second-order linear system is asymptotically stable if and only if $k_{3}^{3}, k_{4}^{3}>0$.
From this point, the proof is similar to the proof of Theorem 3.3.


Figure 3 - Simulation result for a ground target, $d^{*}=2$


Figure 4 - Simulation result for a ground target, $d^{*}=10$


Figure 5 - Simulation result for a frontal target, $d^{*}=2$


Figure 6 - Simulation result for a frontal target, $d^{*}=10$


Figure 7 - Simulation result with constant wind, without wind rejection action, $d^{*}=1$


Figure 8 - Simulation result with constant wind, with wind rejection action, $d^{*}=1$


Figure 9 - Simulation result with varying wind, without wind rejection action, $d^{*}=1$


Figure 10 - Simulation result with varying wind, with wind rejection action, $d^{*}=1$

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[^0]:    2. This author has been supported by the "Chaire d'excellence en Robotique RTE-UPMC"
