Gradient-like observer design on the Special Euclidean group SE(3) with system outputs on the real projective space

Minh-Duc Hua, Tarek Hamel, Robert Mahony, Jochen Trumpf

***Abstract*— A nonlinear observer on the Special Euclidean group $\mathrm{SE}(3)$ for full pose estimation, that takes the system outputs on the real projective space directly as inputs, is proposed. The observer derivation is based on a recent advanced theory on nonlinear observer design. A key advantage with respect to existing pose observers on $\mathrm{SE}(3)$ is that we can now incorporate in a unique observer different types of measurements such as vectorial measurements of known inertial vectors and position measurements of known feature points. The proposed observer is extended allowing for the compensation of unknown constant bias present in the velocity measurements. Rigorous stability analyses are equally provided. Excellent performance of the proposed observers are shown by means of extensive simulations.**

I. Introduction

The development of a robust and reliable estimator of the pose (i.e. position and attitude) of a rigid body is a key requirement for robust and high performance control of robotic vehicles. Pose estimation is a highly nonlinear problem in which the sensors normally utilized are prone to non-Gaussian noise [7]. Classical approaches for state estimation are based on nonlinear filtering techniques such as extended Kalman filters, unscented Kalman filters or particle filters. Recently, nonlinear observers have become an alternative to these classical techniques, starting with the work of Salcudean [19] for attitude estimation and then over the last two decades [16], [24], [17], [21], [20], [13], [2], [23], [15], [18], [8], [9], [6]. Most early nonlinear attitude observers were developed on the basis of Lyapunov analysis. However, the attitude estimation problem has become an intuitive example for the development of recent theories on invariant observers for systems with symmetry [1], [4], [5], [13], [12], [22], [14], [11]. For the attitude estimation problem, Mahony et al. [13] derived a complementary nonlinear attitude observer exploiting the underlying Lie group structure of the Special Orthogonal group $\mathrm{SO}(3)$, and proved almost global stability of the error system. A locally valid symmetry-preserving nonlinear observer design based on the Cartan moving-frame method was proposed in [4], [5]. This process is valid for arbitrary Lie groups but specializes to the same attitude filter on $\mathrm{SO}(3)$. Lageman et al. [12] proposed a gradient-like observer design technique for invariant systems on Lie groups. This method leads to almost globally convergent observers provided that a non-degenerate Morse-Bott cost function is used. More recently, Mahony et al. [14] proposed an observer design method directly on the output space for equivariant kinematics of mechanical systems. The proposed observer structure combined with an equivariant innovation term leads to autonomous error evolution. Moreover, a Lyapunov function construction is used to design the observer innovation in order to ensure the required equivariance and, subsequently, lead to strong convergence properties of the error dynamics. Khosravian et al. [11] extends the observer design methodology proposed [14] for invariant systems on Lie group with outputs on homogeneous spaces and where the measurement of system input is corrupted by an unknown constant bias. It is showed that any candidate observer results in non-autonomous error dynamics, except Abelian Lie groups.

In fact, full pose observer design, although less studied than attitude observer design, has recently attracted more attention [24], [17], [7], [3], [2], [23], [10]. Baldwin et al. proposed observers directly on $\mathrm{SE}(3)$ using both full state feedback [3] and bearing only measurements of known landmarks [2]. Vasconcelos et al. [23] proposed an observer on $\mathrm{SO}(3)\times\mathbb{R}^3$ that uses full range and bearing measurements of known landmarks, achieving almost global asymptotic stability. In a prior work by the authors [10], a nonlinear observer on $\mathrm{SE}(3)$ was proposed using directly position measurements in the body-fixed frame of known inertial feature points or landmarks, with motivation strongly related to robotic vision applications using either stereo camera or Kinect sensor. The observer derivation is based on the gradient-like observer design technique proposed in [12], and the almost global asymptotic stability of the error system is proved by means of Lyapunov analysis.

In this paper, we consider the question of deriving a nonlinear observer on $\mathrm{SE}(3)$ for full pose estimation that takes the system outputs on the real projective space $\mathbb{RP}^3$ directly as inputs. A key advance on our prior work [10] is the possibility of incorporating "naturally" in a sole observer both vectorial measurements (provided e.g. by magnetometers or inclinometers) and position measurements of known inertial feature points (provided e.g. by stereo camera). In addition, sharing the same robustness property with the observer proposed in [10], the algorithm here proposed is also well-posed even when there is insufficient data for full pose reconstruction using algebraic techniques. In such situations, the proposed observer continues to operate, incorporating what information is available and relying on propagation of prior estimates where necessary. Finally, as a complementary

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contribution, a modified version of the basic observer is proposed so as to deal with the case where bias is present in the velocity measurements.

The remainder of this paper is organised as follows. Section II formally introduces the problem of pose estimation on $\mathrm{SE}(3)$ along with the notation used. In Section III, based on a recent advanced theory for nonlinear observer design directly on the output space [14], a nonlinear observer on $\mathrm{SE}(3)$ is proposed using direct body-fixed measurements of known inertial elements of the real projective space $\mathbb{RP}^3$ and the knowledge of the group velocity. Stability analysis is also provided in this section. Then, in Section IV the proposed basic observer is extended using Lyapunov theory in order to cope with the case where the measurement data of the group velocity are corrupted by an unknown constant bias. In Section V, the performance of the proposed observers are validated by means of simulation. Finally, concluding remarks are given in Section VI.

II. Preliminary material

A. Notation

Let $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ denote an inertial frame and a body-fixed frame attached to a vehicle moving in 3D-space, respectively. The vehicle's position, expressed in the frame $\{\mathcal{A}\}$, is denoted as $p \in \mathbb{R}^3$. The attitude of the vehicle is represented by a rotation matrix $R \in \mathrm{SO}(3)$ of the frame $\{\mathcal{B}\}$ relative to the frame $\{\mathcal{A}\}$. Let $V \in \mathbb{R}^3$ denote the vehicle's translational velocity, expressed in $\{\mathcal{B}\}$. Let $\Omega \in \mathbb{R}^3$ denote the vehicle's angular velocity, expressed in $\{\mathcal{B}\}$, of the frame $\{\mathcal{B}\}$ relative to the frame $\{\mathcal{A}\}$.

In this paper, we consider the problem of estimating the vehicle's pose, which can be represented by an element of the Special Euclidean group $\mathrm{SE}(3)$ given by the matrix

$$X := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3) \subset \mathbb{R}^{4\times 4}. \tag{1}$$

This representation, known as homogeneous coordinates, preserves the group structure of $\mathrm{SE}(3)$ with the $\mathrm{GL}(4)$ operation of matrix multiplication, i.e. $X_1X_2 \in \mathrm{SE}(3)$, $\forall X_1, X_2 \in \mathrm{SE}(3)$. Now let us recall some common definitions and notation.

• The Lie-algebra $\mathfrak{se}(3)$ of the group SE(3) is defined as

$$\mathfrak{se}(3) := \left\{ A \in \mathbb{R}^{4\times 4} \mid \exists\, \Omega, V \in \mathbb{R}^3 \,:\, A = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix} \right\},$$

with $\Omega_\times$ denoting the skew-symmetric matrix associated with the cross product by Ω, i.e. $\Omega_\times v = \Omega \times v$, $\forall v \in \mathbb{R}^3$. The adjoint operator is a mapping $Ad : \mathrm{SE}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ defined as $Ad_X A := XAX^{-1}$, with $X \in \mathrm{SE}(3), A \in \mathfrak{se}(3)$.

• For any two matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$, the Euclidean matrix inner product and Frobenius norm are defined as

$$\langle M_1, M_2 \rangle := \mathrm{tr}(M_1^\top M_2), \;\; \|M_1\| := \sqrt{\langle M_1, M_1 \rangle}.$$

Let $\mathbf{P}_a(M)$, $\forall M \in \mathbb{R}^{n\times n}$, denote the anti-symmetric part of M, i.e. $\mathbf{P}_a(M) := (M - M^\top)/2$. Let $\mathbf{P} : \mathbb{R}^{4\times 4} \to \mathfrak{se}(3)$ denote the unique orthogonal projection of $\mathbb{R}^{4\times 4}$ onto $\mathfrak{se}(3)$ with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e. $\forall A \in \mathfrak{se}(3)$, $M \in \mathbb{R}^{4\times 4}$, one has

$$\langle A, M \rangle = \langle A, \mathbf{P}(M) \rangle = \langle \mathbf{P}(M), A \rangle \,.$$

It is verified that for all $M_1 \in \mathbb{R}^{3\times 3}, m_{2,3} \in \mathbb{R}^3, m_4 \in \mathbb{R}$,

$$\mathbf{P}\left(\begin{bmatrix} M_1 & m_2 \\ m_3^\top & m_4 \end{bmatrix} \right) := \begin{bmatrix} \mathbf{P}_a(M_1) & m_2 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

• For all $X \in \mathrm{SE}(3)$, $A_1, A_2 \in \mathfrak{se}(3)$, the following equation defines a *right-invariant Riemannian metric* $\langle \cdot, \cdot \rangle_X$:

$$\langle A_1X, A_2X \rangle_X := \langle A_1, A_2 \rangle.$$

• For any $x \in \mathbb{R}^4$ (or $\in \mathbb{RP}^3$), the notation $\underline{x} \in \mathbb{R}^3$ denotes the vector of first three components of x and the notation x_i stands for the i-th component of x. Thus, it can be written as $x = [\underline{x} \quad x_4]^\top$.

B. System equations and measurements

The vehicle's pose $X \in$ SE(3), defined by (1), satisfies the kinematic equation

$$\dot{X} = F(X, A) := XA, \tag{3}$$

with group velocity $A \in \mathfrak{se}(3)$. System (3) is *left invariant* in the sense that it preserves the (Lie group) invariance properties with respect to constant translation and constant rotation of the body-fixed frame $\{\mathcal{B}\}$ $X \mapsto X_0X$.

Assume that the group velocity A (i.e. Ω and V) is bounded, continuous, and available to measurement. Moreover, $N \in \mathbb{N}^+$ constant elements of the real projective space $\mathring{y}_i \in \mathbb{RP}^3$ $(i = 1, \cdots, N)$, known in the inertial frame $\{\mathcal{A}\}$, are assumed to be measured in the body-fixed frame $\{\mathcal{B}\}$ as

$$y_i = h(X, \mathring{y}_i) := \frac{X^{-1}\mathring{y}_i}{|X^{-1}\mathring{y}_i|} \in \mathbb{RP}^3, \quad i = 1, \cdots, N. \tag{4}$$

Note that the Lie group action $h : \mathrm{SE}(3) \times \mathbb{RP}^3 \to \mathbb{RP}^3$ is transitive and is a *right group action* in the sense that for all $X_1, X_2 \in \mathrm{SE}(3)$ and $y \in \mathbb{RP}^3$, one has $h(X_2, h(X_1, y)) = h(X_1X_2, y)$. For later use, define

$$Y := (y_1, \cdots, y_N), \quad \mathring{Y} := (\mathring{y}_1, \cdots, \mathring{y}_N). \tag{5}$$

Remark 1 *Interestingly, by considering the measurement data in the real projective space $\mathbb{RP}^3$, we are able to combine in a unique pose observer various types of measurements that are provided by sensors of different nature. For instance, from a stereo camera or a Kinect sensor we can obtain a matching of $N_1 \in \mathbb{N}$ feature points whose position coordinates are known in both the inertial reference frame $\{\mathcal{A}\}$ and the current body-fixed frame $\{\mathcal{B}\}$, i.e. one has*

$$p_i = R^\top(\mathring{p}_i - p), \quad i = 1, \cdots, N_1,$$

with $\mathring{p}_i, p_i \in \mathbb{R}^3$ the position coordinates of the feature points expressed in the frames $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$, respectively. Then, the following simple transformations:

$$\underline{\mathring{y}}_i := \frac{\mathring{p}_i}{\sqrt{|\mathring{p}_i|^2 + 1}}, \quad \mathring{y}_{i,4} := \frac{1}{\sqrt{|\mathring{p}_i|^2 + 1}},$$
$$\underline{y}_i := \frac{p_i}{\sqrt{|p_i|^2 + 1}}, \quad y_{i,4} := \frac{1}{\sqrt{|p_i|^2 + 1}},$$

yield the following relations in the form (4)*:*

$$y_i = \frac{X^{-1}\mathring{y}_i}{|X^{-1}\mathring{y}_i|} = h(X, \mathring{y}_i), \quad i = 1, \cdots, N_1,$$

with $\mathring{y}_i = [\underline{\mathring{y}}_i \quad \mathring{y}_{i,4}]^\top \in \mathbb{RP}^3$ *and* $y_i = [\underline{y}_i \quad y_{i,4}]^\top \in \mathbb{RP}^3$*. On the other hand, assume also that the vehicle is equipped with* $N_2 \in \mathbb{N}$ *vectorial sensors (e.g. magnetometer or inclinometer) so as to provide the measurements* $v_j \in \mathbb{R}^3$ *in the body-fixed frame* $\{\mathcal{B}\}$ *of* N_2 *Euclidean vectors (given for example by the geomagnetic field or the gravity field) whose coordinates* $\mathring{v}_j \in \mathbb{R}^3$ *in the inertial frame* $\{\mathcal{A}\}$ *are known. Then, one verifies that* $v_j = R^\top \mathring{v}_j$ *and deduces the following relations in the form* (4)*:*

$$y_j = \frac{X^{-1}\mathring{y}_j}{|X^{-1}\mathring{y}_j|} = h(X, \mathring{y}_j), \quad j = N_1+1, \cdots, N_1+N_2,$$

with $\mathring{y}_j = [\underline{\mathring{y}}_j \quad 0]^\top \in \mathbb{RP}^3$, $y_j = [\underline{y}_j \quad 0]^\top \in \mathbb{RP}^3$, $\underline{\mathring{y}}_j := \frac{\mathring{v}_j}{|\mathring{v}_j|}$ *and* $\underline{y}_j := \frac{v_j}{|v_j|}$.

We verify that $\mathrm{SE}(3)$ is a symmetry group with group actions $\phi : \mathrm{SE}(3) \times \mathrm{SE}(3) \longrightarrow \mathrm{SE}(3)$, $\psi : \mathrm{SE}(3) \times \mathfrak{se}(3) \longrightarrow \mathfrak{se}(3)$ and $\rho : \mathrm{SE}(3) \times \mathbb{RP}^3 \longrightarrow \mathbb{RP}^3$ defined by

$$\begin{array}{rcl} \phi(Q, X) & := & XQ, \\ \psi(Q, A) & := & Ad_{Q^{-1}}A = Q^{-1}AQ, \\ \rho(Q, y) & := & \frac{Q^{-1}y}{|Q^{-1}y|}. \end{array}$$

Indeed, it is straightforward to verify that ϕ, ψ, and ρ are *right group actions* in the sense that $\phi(Q_2, \phi(Q_1, X)) = \phi(Q_1Q_2, X)$, $\psi(Q_2, \psi(Q_1, A)) = \psi(Q_1Q_2, A)$, and $\rho(Q_2, \rho(Q_1, y)) = \rho(Q_1Q_2, y)$, for all $Q_1, Q_2, X \in \mathrm{SE}(3)$, $A \in \mathfrak{se}(3)$, and $y \in \mathbb{RP}^3$. Clearly, one has

$$\rho(Q, h(X, \mathring{y}_i)) = \frac{Q^{-1}\frac{X^{-1}\mathring{y}_i}{|X^{-1}\mathring{y}_i|}}{\left|Q^{-1}\frac{X^{-1}y_i}{|X^{-1}\mathring{y}_i|}\right|} = h(\phi(Q, X), \mathring{y}_i),$$

and

$$\begin{array}{rcl} d\phi_Q(X)[F(X, A)] & = & XAQ = (XQ)(Q^{-1}AQ) \\ & = & F(\phi(Q, X), \psi(Q, A)). \end{array}$$

Thus, the kinematics (3) are *right equivariant* in the sense of [14, Def. 2]. This is a condition allowing us to apply the theory proposed in [14] for nonlinear observer design directly on the output space. Note also that the system under consideration belongs to type I systems (see [14]) where both the velocity sensors and the state sensors are attached to the body-fixed frame.

III. Gradient-like observer design

Denote by $\hat{X}(t) \in \mathrm{SE}(3)$ the estimate of the pose $X(t)$ and denote by $\hat{R}$ and $\hat{p}$ the estimates of R and p, respectively. One has $\hat{X} = \begin{bmatrix} \hat{R} & \hat{p} \\ 0 & 1 \end{bmatrix}$. Define the group error

$$E(\hat{X}, X) := \hat{X}X^{-1} \in \mathrm{SE}(3), \tag{6}$$

which is *right invariant* in the sense that for all $\hat{X}, X, Q \in \mathrm{SE}(3)$, one has $E(\hat{X}Q, XQ) = E(\hat{X}, X)$. From now on, without confusion the shortened notation E is used for $E(\hat{X}, X)$. The group error E converges to the identity element $I_4 \in \mathrm{SE}(3)$ iif $\hat{X}$ converges to X. For later use, define also the output errors $e_i \in \mathbb{RP}^3$, with $i = 1, \cdots, N$, as

$$e_i := h(\hat{X}^{-1}, y_i) = \frac{\hat{X}y_i}{|\hat{X}y_i|} = \frac{E\mathring{y}_i}{|E\mathring{y}_i|}. \tag{7}$$

Note that e_i $(i = 1, \cdots, N)$ can be viewed as the estimates of $\mathring{y}_i$, since they converge to $\mathring{y}_i$ when E converges to I_4. Note also that e_i are computable by the observer.

We now proceed the observer design. As proposed by [14], the observer takes the form

$$\dot{\hat{X}} = \hat{X}A - \Delta(\hat{X}, Y)\hat{X}, \quad \hat{X}(0) \in \mathrm{SE}(3), \tag{8}$$

where $\Delta(\hat{X}, Y) \in \mathfrak{se}(3)$, which is a matrix-valued function of $\hat{X}$ and Y with Y defined by (5), is the innovation term to be designed hereafter and must be *right equivariant* in the sense that $\forall Q \in \mathrm{SE}(3)$:

$$\Delta(\phi(Q, \hat{X}), \rho(Q, Y)) = \Delta(\hat{X}, Y).$$

with $\rho(Q, Y) := (\rho(Q, y_1), \cdots, \rho(Q, y_N))$. Interestingly, if the innovation term $\Delta(\hat{X}, Y)$ is right equivariant, the dynamics of the group error E are autonomous [14, Th. 1]:

$$\dot{E} = -\Delta(E, \mathring{Y})E. \tag{9}$$

In order to determine the innovation term $\Delta(\hat{X}, Y)$, the following cost function is considered:

$$\begin{aligned} &\mathcal{C} : \mathrm{SE}(3) \times (\mathbb{RP}^3 \times \cdots \times \mathbb{RP}^3) \longrightarrow \mathbb{R}^+, \\ &(\hat{X}, Y) \quad \mapsto \mathcal{C}(\hat{X}, Y) := \sum_{i=1}^{N} \frac{k_i}{2}\left|\frac{\hat{X}y_i}{|\hat{X}y_i|} - \mathring{y}_i\right|^2 \end{aligned} \tag{10}$$

with positive constant parameters k_i. It is easily verified that the cost function $\mathcal{C}(\hat{X}, Y)$ is *right invariant* in the sense that $\mathcal{C}(\phi(Q, \hat{X}), \rho(Q, Y)) = \mathcal{C}(\hat{X}, Y)$ for all $Q \in \mathrm{SE}(3)$. From here, the innovation term $\Delta(\hat{X}, Y)$ is computed as [14, Eq. (40)]:

$$\Delta(\hat{X}, Y) := (\mathrm{grad}_1 \mathcal{C}(\hat{X}, Y))\hat{X}^{-1}, \tag{11}$$

where grad_1 is the *gradient* in the first variable, using a right-invariant Riemannian metric on $\mathrm{SE}(3)$.

Lemma 1 *The innovation term* $\Delta(\hat{X}, Y)$ *defined by* (11) *is right equivariant and explicitly given by*

$$\Delta(\hat{X}, Y) = -\mathbf{P}\left(\sum_{i=1}^{N} k_i \left(I_4 - e_i e_i^\top\right) \mathring{y}_i e_i^\top\right), \tag{12}$$

with e_i *considered as functions of* $\hat{X}$ *and* y_i*, i.e.* $e_i = \frac{\hat{X}y_i}{|\hat{X}y_i|}$.

Proof: The proof for $\Delta(\hat{X}, Y)$ given by (12) to be right equivariant is straightforward. Now, using standard rules for transformations of Riemannian gradients and the fact that the Riemannian metric is right invariant, one obtains

$$\begin{aligned} \mathcal{D}_1\mathcal{C}(\hat{X}, Y)[U\hat{X}] &= \langle \mathrm{grad}_1\mathcal{C}(\hat{X}, Y), U\hat{X}\rangle_X \\ &= \langle \mathrm{grad}_1\mathcal{C}(\hat{X}, Y)\hat{X}^{-1}\hat{X}, U\hat{X}\rangle_X \\ &= \langle \mathrm{grad}_1\mathcal{C}(\hat{X}, Y)\hat{X}^{-1}, U\rangle \\ &= \langle \Delta(\hat{X}, Y), U\rangle, \end{aligned} \tag{13}$$

with some $U \in \mathfrak{se}(3)$. On the other hand, using (10) one deduces

$$\begin{aligned}\mathcal{D}_1\mathcal{C}(\hat{X},Y)[U\hat{X}] &= d_1\mathcal{C}(\hat{X},Y)[U\hat{X}] \\ &= \textstyle\sum_{i=1}^N k_i \left(\frac{\hat{X}y_i}{|\hat{X}y_i|} - \mathring{y}_i\right)^\top \left(I_4 - \frac{(\hat{X}y_i)(\hat{X}y_i)^\top}{|\hat{X}y_i|^2}\right)\frac{(U\hat{X})y_i}{|\hat{X}y_i|} \\ &= \textstyle\sum_{i=1}^N k_i(e_i - \mathring{y}_i)^\top (I_4 - e_i e_i^\top)Ue_i \\ &= \operatorname{tr}\left(\textstyle\sum_{i=1}^N k_i(I_4 - e_ie_i^\top)(e_i - \mathring{y}_i)e_i^\top U^\top\right) \\ &= \left\langle -\textstyle\sum_{i=1}^N k_i(I_4 - e_ie_i^\top)\mathring{y}_ie_i^\top, U\right\rangle \\ &= \left\langle -\mathbf{P}\left(\textstyle\sum_{i=1}^N k_i(I_4 - e_ie_i^\top)\mathring{y}_ie_i^\top\right), U\right\rangle. \end{aligned} \tag{14}$$

Finally, the expression of $\Delta(\hat{X},Y)$ given by (12) is directly obtained from (13) and (14). ■

Using the definition (2) of the projection $\mathbf{P}(\cdot)$, the innovation term $\Delta(\hat{X},Y)$ given by (12) can be rewritten in matrix form as follows:

$$\Delta(\hat{X},Y) = \begin{bmatrix} -\frac{1}{2}\sum_{i=1}^N k_i(\underline{e}_i \times \underline{\mathring{y}}_i)_\times & \sum_{i=1}^N k_i e_{i,4}((e_i{}^\top \mathring{y}_i)\underline{e}_i - \underline{\mathring{y}}_i) \\ 0 & 0 \end{bmatrix} \tag{15}$$

Using (9), (11) and (12), one deduces the error system

$$\begin{aligned}\dot{E} &= -\operatorname{grad}_1\mathcal{C}(E,\mathring{Y}) \\ &= \mathbf{P}\left(\sum_{i=1}^N k_i\left(I_4 - e_ie_i{}^\top\right)\mathring{y}_ie_i{}^\top\right)E\end{aligned} \tag{16}$$

with e_i considered as functions of E and $\mathring{y}_i$, i.e. $e_i = \frac{E\mathring{y}_i}{|E\mathring{y}_i|}$.

For the sake of analysis purposes, the following assumption is introduced.

Assumption 1 *(Observability) The set* $\{\mathring{y}_i \in \mathbb{RP}^3, i = 1,\cdots,N\}$ *satisfies one of the three following cases:*

- *Case 1: There exist two different points* $\mathring{y}_{i_1}$ *and* $\mathring{y}_{i_2}$ *with* $\mathring{y}_{i_1,4} = \mathring{y}_{i_2,4} = 0$ *and one point* $\mathring{y}_{j_1}$ *such that* $\mathring{y}_{j_1,4} \neq 0$.
- *Case 2: There exist one point* $\mathring{y}_{i_1}$ *with* $\mathring{y}_{i_1,4} = 0$ *and two different points* $\mathring{y}_{j_1}$ *and* $\mathring{y}_{j_2}$ *(i.e.,* $\mathring{y}_{j_1} \neq \mathring{y}_{j_2}$*) with* $\mathring{y}_{j_1,4} \neq 0$ *and* $\mathring{y}_{j_2,4} \neq 0$. *Furthermore, the vector* $\underline{\mathring{y}}_{i_1}$ *and the resultant vector* $v_{j_{12}} := \mathring{y}_{j_2,4}\underline{\mathring{y}}_{j_1} - \mathring{y}_{j_1,4}\underline{\mathring{y}}_{j_2}$ *are non-collinear.*
- *Case 3: There exist three different points* $\mathring{y}_{j_1}$, $\mathring{y}_{j_2}$ *and* $\mathring{y}_{j_3}$ *such that* $\mathring{y}_{j_1,4} \neq 0$, $\mathring{y}_{j_2,4} \neq 0$ *and* $\mathring{y}_{j_3,4} \neq 0$. *Furthermore, the resultant vectors* $v_{j_{12}} := \mathring{y}_{j_2,4}\underline{\mathring{y}}_{j_1} - \mathring{y}_{j_1,4}\underline{\mathring{y}}_{j_2}$, $v_{j_{23}} := \mathring{y}_{j_3,4}\underline{\mathring{y}}_{j_2} - \mathring{y}_{j_2,4}\underline{\mathring{y}}_{j_3}$ *and* $v_{j_{31}} := \mathring{y}_{j_1,4}\underline{\mathring{y}}_{j_3} - \mathring{y}_{j_3,4}\underline{\mathring{y}}_{j_1}$ *are not all collinear.*

From here, the first result of this paper is stated.

Theorem 1 *Consider the kinematics* (3). *Consider the observer* (8) *with the innovation term* $\Delta(\hat{X},Y)$ *given by* (12). *Assume that Assumption 1 is satisfied. Then, the equilibrium* $E = I_4$ *of the error system* (16) *is locally asymptotically stable.*

Proof: Since the right-hand side of (16) is a gradient flow of $\mathcal{C}$, in order to prove the local asymptotic stability of $E = I_4$, it suffices to prove that $\mathcal{C}(E,\mathring{Y})$ is minimal when $E = I_4$. Note that

$$\mathcal{C}(E,\mathring{Y}) = \mathcal{V}(E) := \frac{1}{2}\sum_{i=1}^N k_i\left|\frac{E\mathring{y}_i}{|E\mathring{y}_i|} - \mathring{y}_i\right|^2. \tag{17}$$

Let us prove that the function $\mathcal{V}(E)$ has a unique global minimum at $E = I_4$, i.e.

$$\mathcal{V}(E) = 0 \Leftrightarrow E = I_4.$$

First, it is straightforward to verify that $\mathcal{V}(I_4) = 0$. Denote $E = \begin{bmatrix} R_e & p_e \\ 0 & 1\end{bmatrix}$, with $R_e \in \mathrm{SO}(3), p_e \in \mathbb{R}^3$. Now assuming that $\mathcal{V}(E) = 0$, we only have to prove that $E = I_4$ or, equivalently, $R_e = I_3$ and $p_e = 0$. In view of (17) and $\mathcal{V}(E) = 0$, one deduces that $E\mathring{y}_i = |E\mathring{y}_i|\mathring{y}_i$, $\forall i$, i.e.

$$\begin{cases} R_e\underline{\mathring{y}}_i + p_e\mathring{y}_{i,4} = \sqrt{\mathring{y}_{i,4}^2 + |R_e\underline{\mathring{y}}_i + p_e\mathring{y}_{i,4}|^2}\;\underline{\mathring{y}}_i & \text{(18a)} \\ \mathring{y}_{i,4} = \sqrt{\mathring{y}_{i,4}^2 + |R_e\underline{\mathring{y}}_i + p_e\mathring{y}_{i,4}|^2}\;\mathring{y}_{i,4} & \text{(18b)}\end{cases}$$

Let us consider all the three cases of Assumption 1.

• *Case 1 of Assumption 1:* Since $\mathring{y}_{i_1,4} = \mathring{y}_{i_2,4} = 0$, one has $|\underline{\mathring{y}}_{i_1}| = |\underline{\mathring{y}}_{i_2}| = 1$. Then, one deduces from (18a) that $R_e\underline{\mathring{y}}_{i_1} = \underline{\mathring{y}}_{i_1}$ and $R_e\underline{\mathring{y}}_{i_2} = \underline{\mathring{y}}_{i_2}$. These equalities and the non-collinearity of $\underline{\mathring{y}}_{i_1}$ and $\underline{\mathring{y}}_{i_2}$ allows one to deduce that $R_e = I_3$. Since $\mathring{y}_{j_1,4} \neq 0$, (18b) implies that $|E\mathring{y}_{j_1}| = 1$. As a consequence, one deduces from (18a) that $p_e = 0$.

• *Case 2 of Assumption 1:* Analogously to case 1, one deduces that $R_e\underline{\mathring{y}}_{i_1} = \underline{\mathring{y}}_{i_1}$. Now, since $\mathring{y}_{j_1,4} \neq 0$ and $\mathring{y}_{j_2,4} \neq 0$, (18b) implies that $|E\mathring{y}_{j_1}| = |E\mathring{y}_{j_2}| = 1$. Then, from (18a) one obtains

$$\begin{cases}(R_e - I_3)\underline{\mathring{y}}_{j_1} + p_e\mathring{y}_{j_1,4} = 0 \\ (R_e - I_3)\underline{\mathring{y}}_{j_2} + p_e\mathring{y}_{j_2,4} = 0\end{cases}$$

From here, simple combination yields $R_ev_{j_{12}} = v_{j_{12}}$, with $v_{j_{12}}$ defined in Assumption 1. One easily verifies that $v_{j_{12}} \neq 0$ using the fact that $\mathring{y}_{j_1}$ and $\mathring{y}_{j_2}$ are non-collinear by assumption. Furthermore, since $\underline{\mathring{y}}_{i_1}$ and $v_{j_{12}}$ are non-collinear by assumption, relations $R_e\underline{\mathring{y}}_{i_1} = \underline{\mathring{y}}_{i_1}$ and $R_ev_{j_{12}} = v_{j_{12}}$ obtained previously imply that $R_e = I_3$. From here, it is straightforward to deduce that $p_e = 0$.

• *Case 3 of Assumption 1:* Analogously to case 2, one deduces from (18) that $|E\mathring{y}_{j_1}| = |E\mathring{y}_{j_2}| = |E\mathring{y}_{j_3}| = 1$ and

$$\begin{cases}(R_e - I_3)\underline{\mathring{y}}_{j_1} + p_e\mathring{y}_{j_1,4} = 0 \\ (R_e - I_3)\underline{\mathring{y}}_{j_2} + p_e\mathring{y}_{j_2,4} = 0 \\ (R_e - I_3)\underline{\mathring{y}}_{j_3} + p_e\mathring{y}_{j_3,4} = 0\end{cases}$$

From here, analogously to case 2 one deduces that $R_ev_{j_{12}} = v_{j_{12}}$, $R_ev_{j_{23}} = v_{j_{23}}$, $R_ev_{j_{31}} = v_{j_{31}}$, and that $v_{j_{12}}$, $v_{j_{23}}$ and $v_{j_{31}}$ are not null. Then, using the non-collinearity assumption of the vectors $v_{j_{12}}$, $v_{j_{23}}$ and $v_{j_{31}}$, one easily deduces that $R_e = I_3$ and, consequently, that $p_e = 0$. ■

IV. Observer design with velocity bias compensation

In this section, the observer developed in the previous section will be extended in order to cope with the case where the measurement $A_y \in \mathfrak{se}(3)$ of the group velocity $A \in \mathfrak{se}(3)$ is corrupted by an unknown constant bias $b_A \in \mathfrak{se}(3)$, i.e.

$$A_y = A + b_A\,.$$

Assumption 2 *Assume that the following matrices $\mathring{G} \in \mathbb{R}^{3\times 3}$ and $\mathring{H} \in \mathbb{R}^{3\times 3}$ are full rank:*

$$\begin{aligned}\mathring{G} &:= \sum_{i=1}^{N} k_i (\underline{\mathring{y}}_{i\times})^2 \\ \mathring{H} &:= \left(\sum_{i=1}^{N} k_i \mathring{y}_{i,4} \underline{\mathring{y}}_{i\times}\right) \mathring{G}^{-1} \left(\sum_{i=1}^{N} k_i \mathring{y}_{i,4} \underline{\mathring{y}}_{i\times}\right) \\ &\quad - \sum_{i=1}^{N} k_i \mathring{y}_{i,4}^2 (I_3 - \underline{\mathring{y}}_i \underline{\mathring{y}}_i^\top)\end{aligned}$$

The condition on the set $\{\mathring{y}_i \in \mathbb{RP}^3, i=1,\cdots,N\}$ evoked in Assumption 1 ensures that it is always possible to choose a set of parameters $\{k_i, i=1,\cdots,N\}$ such that $\mathring{G}$ and $\mathring{H}$ are full rank (i.e. invertible). Now, the second result of this paper is stated.

Proposition 1 *Consider the observer system*

$$\begin{cases}\dot{\hat{X}} = \hat{X}(A_y - \hat{b}_A) - \Delta(\hat{X}, Y)\hat{X} & \text{(21a)} \\ \dot{\hat{b}}_A = -k_b \mathbf{P}\left(\hat{X}^\top \sum_{i=1}^{N} k_i \left(I_4 - e_i e_i{}^\top\right) \mathring{y}_i e_i{}^\top \hat{X}^{-\top}\right) & \text{(21b)} \\ \hat{X}(0) \in \mathrm{SE}(3),\;\; \hat{b}_A(0) \in \mathfrak{se}(3)\end{cases}$$

with $\Delta(\hat{X}, Y)$ given by (12)*. Assume that Assumptions 1 and 2 are satisfied. Assume also that A and X are bounded for all time. Then, the equilibrium $(E, \tilde{b}_A) = (I_4, 0)$ of the dynamics of $(E, \tilde{b}_A)$, with $\tilde{b}_A := b_A - \hat{b}_A$, is locally asymptotically stable.*

Proof: It is easily verified that $\dot{\tilde{b}}_A = -\dot{\hat{b}}_A$ and $\dot{\hat{X}} = \hat{X}(A + \tilde{b}_A) - \Delta(\hat{X}, Y)\hat{X}$. Then, one deduces

$$\dot{E} = \left(Ad_{\hat{X}} \tilde{b}_A - \Delta(E, \mathring{Y})\right) E. \tag{22}$$

Now, consider the candidate Lyapunov function

$$\mathcal{V}_b(E, \tilde{b}_A) := \frac{1}{2} \sum_{i=1}^{N} k_i \left| \frac{E\mathring{y}_i}{|E\mathring{y}_i|} - \mathring{y}_i \right|^2 + \frac{1}{2k_b} \|\tilde{b}_A\|^2. \tag{23}$$

Analogously to the proof of Theorem 1, it can be verified that $\mathcal{V}_b(E, \tilde{b}_A)$ is locally positive-definite and has a unique global minimum at $(E, \tilde{b}_A) = (I_4, 0)$, i.e. $\mathcal{V}_b(E, \tilde{b}_A) = 0 \Leftrightarrow (E, \tilde{b}_A) = (I_4, 0)$.

The time-derivative of $\mathcal{V}_b$ satisfies

$$\begin{aligned}\dot{\mathcal{V}}_b &= \left\langle -\textstyle\sum_{i=1}^{N} k_i \left(I_4 - e_i e_i{}^\top\right) \mathring{y}_i e_i{}^\top, Ad_{\hat{X}} \tilde{b}_A - \Delta(E, \mathring{Y}) \right\rangle \\ &\quad - \tfrac{1}{k_b} \left\langle \dot{\hat{b}}_A, \tilde{b}_A \right\rangle \\ &= -\left\| \mathbf{P}\left(\textstyle\sum_{i=1}^{N} k_i \left(I_4 - e_i e_i^\top\right) \mathring{y}_i e_i^\top\right) \right\|^2 \\ &= -\|\Delta(E, \mathring{Y})\|^2.\end{aligned} \tag{24}$$

Since the dynamics of $(E, \tilde{b}_A)$ are not autonomous, LaSalle's theorem does not apply to deduce the convergence of $\dot{\mathcal{V}}_b$ to zero. Thus, the next step of the proof consists in proving that $\dot{\mathcal{V}}_b$ is (locally) uniformly continuous along every system's solution in order to deduce, by application of Barbalat's lemma, the convergence of $\dot{\mathcal{V}}_b$ to zero. To this purpose it suffices to prove that $\ddot{\mathcal{V}}_b$ is bounded. In view of (24), $\ddot{\mathcal{V}}_b$ is bounded if $\dot{e}_i$ $(i=1,\cdots,N)$ are bounded, where (using (22) and the relation $e_i = \frac{E\mathring{y}_i}{|E\mathring{y}_i|}$)

$$\dot{e}_i = (I_4 - e_i e_i^\top)(Ad_{\hat{X}} \tilde{b}_A - \Delta(E, \mathring{Y}))e_i.$$

According to Assumption 2, there exists at least one point $\mathring{y}_i$ such that its fourth component $\mathring{y}_{i,4}$ is not null. This indicates that for a given small number $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|p_e| > \delta_\varepsilon$ or $|\tilde{b}_A| > \delta_\varepsilon$ then $\mathcal{V}_b(E, \tilde{b}_A) > \varepsilon$. Therefore, there exists a small enough neighborhood $\mathfrak{B}_\varepsilon \in \mathrm{SE}(3) \times \mathbb{R}^3$ of the point $(I_4, 0)$ such that if $(E(0), \tilde{b}_A(0)) \in \mathfrak{B}_\varepsilon$ then $\mathcal{V}_b(E(0), \tilde{b}_A(0)) < \varepsilon$. Since $\mathcal{V}_b(E, \tilde{b}_A)$ is non-increasing, one has $\mathcal{V}_b(E(t), \tilde{b}_A(t)) < \varepsilon, \forall t \le 0$. This implies that E and $\tilde{b}_A$ remain bounded. Since X is bounded by assumption, one deduces from the boundedness of E that $\hat{X}$ is also bounded, which in turn implies the boundedness of $\dot{E}$ and $\dot{e}_i$. This concludes the proof of (local) uniform continuity of $\dot{\mathcal{V}}_b$ and the convergence of $\dot{\mathcal{V}}_b$ to zero. One easily verifies that $(E, \tilde{b}_A) = (I_4, 0)$ is an equilibrium of the error system. Let us prove the local stability of this equilibrium. To this purpose let us first prove that $\forall (E, \tilde{b}_A) \in \mathfrak{B}_\varepsilon$:

$$\begin{cases}\dot{\mathcal{V}}_b(E, \tilde{b}_A) = 0 & \text{if} \quad E = I_4 \\ \dot{\mathcal{V}}_b(E, \tilde{b}_A) < 0 & \text{if} \quad E \neq I_4\end{cases}$$

Consider a first order approximation of $E = \begin{bmatrix} R_e & p_e \\ 0 & 1 \end{bmatrix}$ around I_4 as

$$\begin{cases} p_e = \varepsilon_p \\ R_e = I_3 + \varepsilon_{r\times}\end{cases}$$

with $\varepsilon_p, \varepsilon_r \in \mathbb{R}^3$. We only need to prove that

$$\dot{\mathcal{V}}_b(E, \tilde{b}_A) = 0 \;\Leftrightarrow\; \varepsilon_p = \varepsilon_r = 0.$$

Note that (24) and (15) indicate that the relation $\dot{\mathcal{V}}_b = 0$ is equivalent to

$$\begin{cases} \sum_{i=1}^{N} k_i \underline{e}_i \times \underline{\mathring{y}}_i = 0 \\ \sum_{i=1}^{N} k_i ((e_i{}^\top \mathring{y}_i)\underline{e}_i - \underline{\mathring{y}}_i) e_{i,4} = 0\end{cases} \tag{25}$$

In first order approximations, one verifies that

$$E\mathring{y}_i = \begin{bmatrix} \underline{\mathring{y}}_i + \varepsilon_{r\times} \underline{\mathring{y}}_i + \mathring{y}_{i,4} \varepsilon_p \\ \mathring{y}_{i,4} \end{bmatrix},$$

$$|E\mathring{y}_i| = 1 + \mathring{y}_{i,4} \varepsilon_p^\top \underline{\mathring{y}}_i,$$

and, thus,

$$e_i = \frac{E\mathring{y}_i}{|E\mathring{y}_i|} = \begin{bmatrix} \underline{\mathring{y}}_i + \varepsilon_{r\times}\underline{\mathring{y}}_i + \mathring{y}_{i,4}(I_3 - \underline{\mathring{y}}_i\underline{\mathring{y}}_i^\top)\varepsilon_p \\ \mathring{y}_{i,4} - \mathring{y}_{i,4}^2\varepsilon_p^\top\underline{\mathring{y}}_i \end{bmatrix}.$$

Therefore, in first order approximations the equalities in (25) can be rewritten as

$$\begin{cases} \left(\sum_{i=1}^N k_i\mathring{y}_{i,4}\underline{\mathring{y}}_{i\times}\right)\varepsilon_p = \left(\sum_{i=1}^N k_i(\underline{\mathring{y}}_{i\times})^2\right)\varepsilon_r & \text{(26a)} \\ \left(\sum_{i=1}^N k_i\mathring{y}_{i,4}^2(I_3 - \underline{\mathring{y}}_i\underline{\mathring{y}}_i^\top)\right)\varepsilon_p = \left(\sum_{i=1}^N k_i\mathring{y}_{i,4}\underline{\mathring{y}}_{i\times}\right)\varepsilon_r & \text{(26b)} \end{cases}$$

Since $\mathring{G}$ is full rank according to Assumption 2, it is deduced from (26a) that $\varepsilon_r = \mathring{G}^{-1}\left(\sum_i k_i\mathring{y}_{i,4}\underline{\mathring{y}}_{i\times}\right)\varepsilon_p$. This relation along with (26b) yields $\mathring{H}\varepsilon_p = 0$. Since $\mathring{H}$ is also full rank by Assumption 2, it is deduced that $\varepsilon_p = 0$ and, consequently, $\varepsilon_r = 0$.

It remains to prove to convergence of $\tilde{b}_A$ to zero. From the convergence of E to I_4 (proven previously) and (22), the application of Barbalat's lemma yields the convergence of $\dot{E}$ to zero. Finally, Eq. (22) and the convergence of $\dot{E}$ and of $\Delta(E,\mathring{Y})$ to zero imply the convergence of $\tilde{b}_A$ to zero. ■

Remark 2 *The estimate $\hat{b}_A$ plays the role of integral correction for the error dynamics* (22)*, allowing for the compensation of the unknown constant bias b_A. It may, however, grow arbitrarily large, resulting in slow convergence and sluggish dynamics of the error evolution. This leads us to replace hereafter the integral term $\hat{b}_A$, with dynamics given by* (21b)*, by an "anti-windup" integrator similar to the one proposed in [9]. More precisely, by decomposing $\hat{b}_A$ as $\hat{b}_A = \begin{bmatrix} (\hat{b}_\Omega)_\times & \hat{b}_V \\ 0 & 0 \end{bmatrix}$ with $\hat{b}_\Omega, \hat{b}_V \in \mathbb{R}^3$, one rewrites the dynamics* (21b) *of the estimated bias $\hat{b}_A$ as*

$$\begin{cases} \dot{\hat{b}}_\Omega = k_b\hat{R}^\top\left(\Omega_\Delta + \frac{1}{2}V_\Delta \times \hat{p}\right) \\ \dot{\hat{b}}_V = k_b\hat{R}^\top V_\Delta \end{cases}$$

with $V_\Delta := \sum_{i=1}^N k_i e_{i,4}((e_i{}^\top\mathring{y}_i)\underline{e}_i - \underline{\mathring{y}}_i)$ and $\Omega_\Delta := -\frac{1}{2}\sum_{i=1}^N k_i\underline{e}_i \times \underline{\mathring{y}}_i$. From here, the following modified dynamics of $\hat{b}_A$ (i.e. of $\hat{b}_\Omega$ and $\hat{b}_V$) are proposed:

$$\begin{cases} \dot{\hat{b}}_\Omega = k_b\hat{R}^\top(\Omega_\Delta + \frac{1}{2}V_\Delta \times \hat{p}) - \kappa_\Omega(\hat{b}_\Omega - \mathrm{sat}_{\delta_\Omega}(\hat{b}_\Omega)) & \text{(28a)} \\ \dot{\hat{b}}_V = k_b\hat{R}^\top V_\Delta - \kappa_V(\hat{b}_V - \mathrm{sat}_{\delta_V}(\hat{b}_V)) & \text{(28b)} \end{cases}$$

with initial conditions $|b_\Omega(0)| \le \delta_\Omega$ and $|b_V(0)| \le \delta_V$; κ_Ω and κ_V two positive numbers; δ_Ω and δ_V two positive parameters associated with the classical functions $\mathrm{sat}_{\delta_\Omega}(\cdot)$ and $\mathrm{sat}_{\delta_\Omega}(\cdot)$ defined by $\mathrm{sat}_\delta(x) = x\min(1, \delta/|x|), \forall x \in \mathbb{R}^3$. The values of δ_Ω and δ_V correspond to initial guesses on the bounds of b_Ω and b_V, i.e. $|b_\Omega| \le \delta_\Omega$ and $|b_V| \le \delta_V$. Then, based on the inequality $|b - \mathrm{sat}_\delta(b - \tilde{b})| \le |\tilde{b}|$ for all $\tilde{b} \in \mathbb{R}^3$ and provided that $\delta \ge |b|$ (see e.g. [9]), it can be easily proved that the time-derivative of $\mathcal{V}_b$ defined by (23) *satisfies $\dot{\mathcal{V}}_b \le -\|\Delta(E,\mathring{Y})\|^2$. Therefore, the convergence and stability properties given in Proposition 1 still holds when the dynamics of $\hat{b}_A$ given by* (21b) *is replaced by* (28)*.*

V. Simulation results

In this section, the performance of observer (21), with (21b) replaces by (28), is illustrated by simulations. The angular and translational velocity measurements are corrupted by the following constant biases:

$$b_\Omega = [-0.02 \quad 0.02 \quad 0.01]^\top \quad (\mathrm{rad/s}),$$
$$b_V = [0.2 \quad -0.1 \quad 0.1]^\top \quad (\mathrm{m/s}).$$

We consider the three following cases where only three system outputs $y_i \in \mathbb{RP}^3$ of known inertial elements $\mathring{y}_i \in \mathbb{RP}^3$ $(i = 1,2,3)$ are available to measurement:

- Case 1: corresponds to Case 1 of Assumption 1, in which two vectorial measurements $v_1, v_2 \in \mathbb{R}^3$ and the position measurement $p_1 \in \mathbb{R}^3$ of one feature point are available, where
$$v_1 = R^\top\mathring{v}_1,\ v_2 = R^\top\mathring{v}_2,\ p_1 = R^\top(\mathring{p}_1 - p),$$
with $\mathring{v}_1 = [0 \quad 0 \quad 1]^\top$, $\mathring{v}_2 = [\sqrt{3}/2 \quad 1/2 \quad 0]^\top$ and $\mathring{p}_1 = [1 \quad 0 \quad 0]^\top$.
- Case 2: corresponds to Case 2 of Assumption 1, in which one vectorial measurement $v_1 \in \mathbb{R}^3$ and the position measurements $p_1, p_2 \in \mathbb{R}^3$ of two feature points are available, where
$$v_1 = R^\top\mathring{v},\ p_1 = R^\top(\mathring{p}_1 - p),\ p_2 = R^\top(\mathring{p}_2 - p),$$
with $\mathring{v}_1 = [0 \quad 0 \quad 1]^\top$, $\mathring{p}_1 = [1 \quad 0 \quad 0]^\top$ and $\mathring{p}_2 = [-1/2 \quad \sqrt{3}/2 \quad 0]^\top$.
- Case 3: corresponds to Case 3 of Assumption 1, in which the position measurements $p_1, p_2, p_3 \in \mathbb{R}^3$ of three feature points are available, where
$$p_1 = R^\top(\mathring{p}_1 - p),\ p_2 = R^\top(\mathring{p}_2 - p),\ p_3 = R^\top(\mathring{p}_3 - p),$$
with $\mathring{p}_1 = [1 \quad 0 \quad 0]^\top$, $\mathring{p}_2 = [-1/2 \quad \sqrt{3}/2 \quad 0]^\top$ and $\mathring{p}_3 = [-1/2 \quad -\sqrt{3}/2 \quad 0]^\top$.

Recall that Remark 1 explains how to transform a vector or a position of a feature point into a corresponding element of $\mathbb{RP}^3$.

The gains and parameters involved in the proposed observer are chosen as follows:

$$k_1 = k_2 = k_3 = 2,\ k_b = 1,$$
$$\kappa_\Omega = \kappa_V = 10,\ \delta_\Omega = 0.052,\ \delta_V = 0.346.$$

For each simulation run, the proposed filter is initialized at the origin (i.e. $\hat{R} = I_3, \hat{p} = 0, \hat{b}_\Omega = 0, \hat{b}_V = 0$) while the true trajectories are initialized differently. Combined sinusoidal inputs are considered for both the angular and translational velocity inputs of the system kinematics. The rotation angle associated with the axis-angle representation is used to represent the attitude trajectory. One can observe from Figure 1 that the observer trajectories converge to the true trajectories after a short transition period for all the three considered cases. Figure 2 shows that the norms of the estimated velocity bias errors $|\tilde{b}_\Omega|$ and $|\tilde{b}_V|$ converge to zero, which means that the group velocity bias b_A is also correctly estimated.

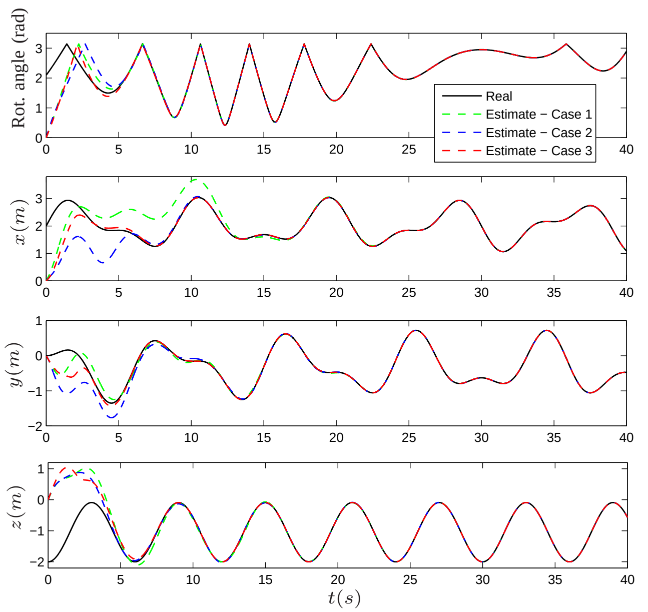

Fig. 1. The rotation angle and the position tracking performance of the proposed observer. Note that the dashed lines are the estimated trajectories (for Cases 1 (green), Case 2 (blue), Case 3 (red)) while the solid line (black) represents the true trajectory.

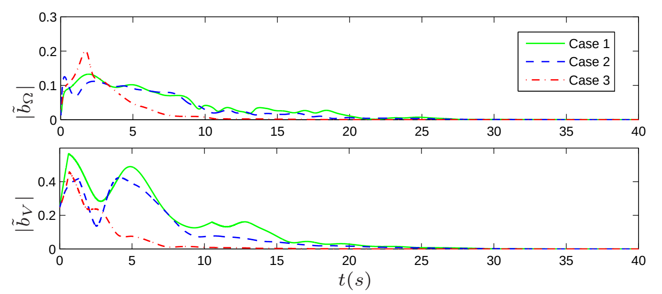

Fig. 2. The norms of the estimated velocity bias errors $|\tilde{b}_\Omega|$ and $|\tilde{b}_V|$ vs. time.

VI. Conclusions

In this paper, we propose a nonlinear observer on $SE(3)$ for full pose estimation that takes the system outputs on the real projective space $\mathbb{RP}^3$ directly as inputs. The observer derivation is based on a recent observer design technique directly on the output space, proposed in [14]. An advantage with respect to our prior work [10] is that we can now incorporate in a unique observer different types of measurements such as vectorial measurements of known inertial vectors and position measurements of known feature points. The proposed observer is also extended on $\mathrm{SE}(3) \times \mathfrak{se}(3)$ so as to compensate for unknown additive constant bias in the velocity measurements. Rigorous stability analyses are equally provided. Excellent performance of the proposed observers are justified through extensive simulations.

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References

[1] N. Aghannan and P. Rouchon. An intrinsic observer for a class of lagrangian systems. *IEEE Trans. on Automatic Control*, 48(6):936–945, 2003.

[2] G. Baldwin, R. Mahony, and J. Trumpf. A nonlinear observer for 6 DOF pose estimation from inertial and bearing measurements. In *IEEE Conf. on Robotics and Automation*, pages 2237–2242, 2009.

[3] G. Baldwin, R. Mahony, J. Trumpf, T. Hamel, and T. Cheviron. Complementary filter design on the Special Euclidean group SE(3). In *European Control Conference*, pages 3763–3770, 2007.

[4] S. Bonnabel, P. Martin, and P. Rouchon. Symmetry-preserving observers. *IEEE Trans. on Automatic Control*, 53(11):2514–2526, 2008.

[5] S. Bonnabel, P. Martin, and P. Rouchon. Non-linear symmetry preserving observers on lie groups. *IEEE Trans. on Automatic Control*, 54(7):1709–1713, 2009.

[6] A. Eudes, P. Morin, R. Mahony, and T. Hamel. Visuo-inertial fusion for homography-based filtering and estimation. In *IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, pages 5186–5192, 2013.

[7] G. D. Baldwin. *Inertial Vision Pose Estimation Using Non-Linear Observers*. PhD thesis, The Australian National University, 2009.

[8] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi. Attitude estimation using biased gyro and vector measurements with time-varying reference vectors. *IEEE Trans. on Automatic Control*, 57(5):1332–1338, 2012.

[9] M.-D. Hua, G. Ducard, T. Hamel, R. Mahony, and K. Rudin. Implementation of a nonlinear attitude estimator for aerial robotic vehicles. *IEEE Trans. on Control Systems Technology*, 22(1):201–213, 2014.

[10] M.-D. Hua, M. Zamani, J. Trumpf, R. Mahony, and T. Hamel. Observer design on the special euclidean group SE(3). In *IEEE Conf. on Decision and Control*, pages 8169–8175, 2011.

[11] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman. Observers for invariant systems on lie groups with biased input measurements and homogeneous outputs. *Automatica*, 55:19–26, 2015.

[12] C. Lageman, J. Trumpf, and R. Mahony. Gradient-like observers for invariant dynamics on a Lie group. *IEEE Trans. on Automatic Control*, 55(2):367–377, 2010.

[13] R. Mahony, T. Hamel, and J.-M. Pflimlin. Nonlinear complementary filters on the Special Orthogonal group. *IEEE Trans. on Automatic Control*, 53(5):1203–1218, 2008.

[14] R. Mahony, J. Trumpf, and T. Hamel. Observers for kinematic systems with symmetry. In *9th IFAC Symposium on Nonlinear Control Systems*, pages 617–633, 2013.

[15] P. Martin and E. Salaun. Design and implementation of a low-cost observer-based attitude and heading reference system. *Control Engineering Practice*, 18(7):712–722, 2010.

[16] H. Nijmeijer and T. I. Fossen. *New Directions in Nonlinear Observer Design*, volume 244 of Lecture Notes in Control and Information Sciences. Springer, 1999.

[17] H. Rehbinder and B. Ghosh. Pose estimation using line-based dynamic vision and inertial sensors. *IEEE Trans. on Automatic Control*, 48(2):186–199, 2003.

[18] A. Roberts and A. Tayebi. On the attitude estimation of accelerating rigid-bodies using GPS and IMU measurements. In *IEEE Conf. on Decision and Control*, pages 8088–8093, 2011.

[19] S. Salcudean. A globally convergent angular velocity observer for rigid body motion. *IEEE Trans. on Automatic Control*, 36(12):1493–1497, 1991.

[20] A. Tayebi, S. McGilvray, A. Roberts, and M. Moallem. Attitude estimation and stabilization of a rigid body using low-cost sensors. In *IEEE Conf. on Decision and Control*, pages 6424–6429, 2007.

[21] J. Thienel and R. M. Sanner. A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise. *IEEE Trans. on Automatic Control*, 48(11):2011–2015, 2003.

[22] J. Trumpf, R. Mahony, T. Hamel, and C. Lageman. Analysis of non-linear attitude observers for time-varying reference measurements. *IEEE Trans. on Automatic Control*, 57:2789–2800, 2012.

[23] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira. A nonlinear position and attitude observer on SE(3) using landmark measurements. *Systems & Control Letters*, 59:155–166, 2010.

[24] B. Vik and T. Fossen. A nonlinear observer for GPS and INS integration. In *IEEE Conf. on Decision and Control*, pages 2956–2961, 2001.